7 The Waveform Channel

The waveform transmitted by the digital demodulator will be corrupted by the channel before it reaches the digital demodulator in the receiver. One important part of the channel is the noise. In continuous time, this random noise is viewed as a random process. So, we first provide some introduction to random processes.

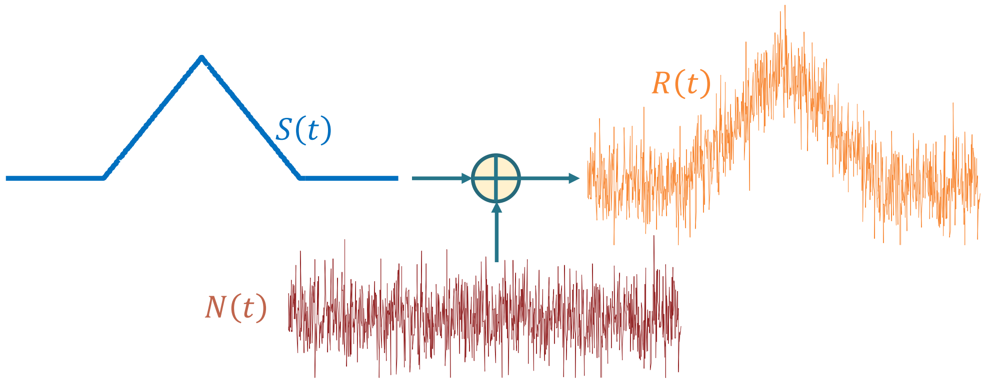


Figure 23: Additive noise channel

7.1 Random Processes

A random process is simply an infinite collection of random variables. These random variables are usually indexed by time. So, the obvious notation for random process would be $X(t)$. As in the signals-and-systems class, time can be discrete or continuous. When time is discrete, it may be more appropriate to use $X_1, X_2, \ldots$ or $X[1], X[2], X[3], \ldots$ to denote a random process.

Example 7.1. Sequence of results (0 or 1) from a sequence of Bernoulli trials is a discrete-time random process.

Example 7.2. Gaussian Random Processes: A random process $X(t)$ is Gaussian if for all positive integers n and for all $t_1, t_2, \ldots, t_n$, the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian random variables.

7.3. For random variable, two important statistics are the mean and the standard deviation (or the variance). For random processes, two important statistics are the mean function and the auto-correlation function (or the power spectral density function).

Definition 7.4. At any particular time t, a random process is simply a random variable and hence we can also find its expected value in the usual way. The **mean function** $m_X(t)$ captures these expected values as a deterministic function of time:

$$m_X(t) = \mathbb{E}\left[X(t)\right].$$

7.1.1 Autocorrelation Function and WSS

One of the most important characteristics of a random process is its autocorrelation function, which leads to the spectral information of the random process. The frequency content process depends on the rapidity of the amplitude change with time. This can be measured by correlating the values of the process at two time instances t_l and t_2.

Definition 7.5. ***Autocorrelation Function***: The autocorrelation function $R_X(t_1, t_2)$ for a random process $X(t)$ is defined by

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1)X(t_2)\right].$$

Example 7.6 (Randomly Phased Sinusoid)**.** Consider a random process

$$X(t) = 5\cos(7t + \Theta)$$

where Θ is a uniform random variable on the interval $(0, 2\pi)$.

$$\begin{aligned} m_X(t) = \mathbb{E}\left[X(t)\right] &= \int_{-\infty}^{+\infty} 5\cos(7t+\theta) f_\Theta(\theta) d\theta \\ &= \int_0^{2\pi} 5\cos(7t+\theta)\frac{1}{2\pi}d\theta = 0. \end{aligned}$$

and

$$\begin{aligned} R_X(t_1, t_2) &= \mathbb{E}\left[X(t_1)X(t_2)\right] \\ &= \mathbb{E}\left[5\cos(7t_1+\Theta) \times 5\cos(7t_2+\Theta)\right] \\ &= \frac{25}{2}\cos\left(7(t_2 - t_1)\right). \end{aligned}$$

Definition 7.7. A random process whose statistical characteristics do not change with time is classified as a ***stationary*** random process. For a stationary process, we can say that a shift of time origin will be impossible to detect; the process will appear to be the same.

Example 7.8. The random process representing the temperature of a city is an example of a ***nonstationary*** process, because the temperature statistics (mean value, for example) depend on the time of the day.

On the other hand, the noise process is stationary, because its statistics (the mean ad the mean square values, for example) do not change with time.

7.9. In general, it is not easy to determine whether a process is stationary. In practice, we can ascertain stationary if there is no change in the signal-generating mechanism. Such is the case for the noise process.

A process may not be stationary in the strict sense. A more relaxed condition for stationary can also be considered.

Definition 7.10. A random process $X(t)$ is ***wide-sense stationary (WSS)*** if

(a) $m_X(t)$ is a constant

(b) $R_X(t_1, t_2)$ depends only on the time difference $t_2 - t_1$ and does not depend on the specific values of t_1 and t_2.

In which case, we can write the correlation function as $R_X(\tau)$ where $\tau = t_2 - t_1$.

- One important consequence is that $\mathbb{E}\left[X^2(t)\right]$ will be a constant as well.

Example 7.11. The random process defined in Example 7.6 is WSS with

$$R_X(\tau) = \frac{25}{2}\cos(7\tau).$$

7.12. Most information signals and noise sources encountered in communication systems are well modeled as WSS random processes.

Example 7.13. White noise process is a WSS process $N(t)$ whose

(a) $\mathbb{E}[N(t)] = 0$ for all t and

(b) $R_N(\tau) = \frac{N_0}{2}\delta(\tau)$.

See also 7.19 for its definition.

- Since $R_N(\tau) = 0$ for $\tau \neq 0$, any two different samples of white noise, no matter how close in time they are taken, are uncorrelated.

Example 7.14. [Thermal noise] A statistical analysis of the random motion (by thermal agitation) of electrons shows that the autocorrelation of thermal noise $N(t)$ is well modeled as

$$R_N(\tau) = kTG\frac{e^{-\frac{\tau}{t_0}}}{t_0} \text{ watts,}$$

where k is Boltzmann's constant ($k = 1.38 \times 10^{-23}$ joule/degree Kelvin), G is the conductance of the resistor (mhos), T is the (ambient) temperature in degrees Kelvin, and t_0 is the statistical average of time intervals between collisions of free electrons in the resistor, which is on the order of 10^{-12} seconds. [11, p. 105]

7.1.2 Power Spectral Density (PSD)

An electrical engineer instinctively thinks of signals and linear systems in terms of their frequency-domain descriptions. Linear systems are characterized by their frequency response (the transfer function), and signals are expressed in terms of the relative amplitudes and phases of their frequency components (the Fourier transform). From the knowledge of the input spectrum and transfer function, the response of a linear system to a given signal can be obtained in terms of the frequency content of that signal. This is an important procedure for deterministic signals. We may wonder if similar methods may be found for random processes.

In the study of stochastic processes, the power spectral density function, $S_X(f)$, provides a frequency-domain representation of the time structure of $X(t)$.

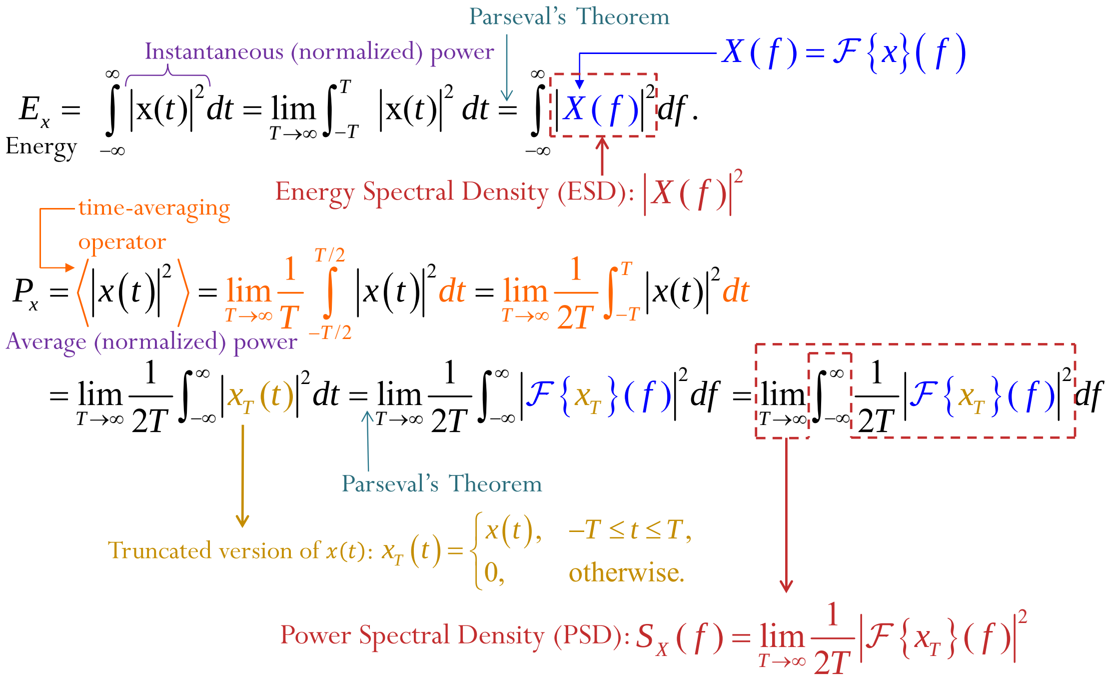


Figure 24: Motivation for the defining formula of the power spectral density for deterministic signal.

You may recall that not all functions of time have Fourier transforms. For many functions that extend over infinite time, the Fourier transform does not exist. Sample functions $x(t)$ of a stationary stochastic process $X(t)$ are usually of this nature. To work with these functions in the frequency domain, we consider $X_T(t)$ which is the **truncated** version of $X(t)$. It is identical to $X(t)$ for $-T \le t \le T$ and 0 elsewhere:

$$X_T(t) = \left\{ \begin{array}{ll} X(t), & -T \le t \le T, \\ 0, & \text{otherwise.} \end{array} \right\} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \mathcal{F}\{X_T\}(f)$$

We use $\mathcal{F}\{X_T\}(f)$ to represent the Fourier transform of $X_T(t)$ evaluated at the frequency f.

Definition 7.15. Consider a WSS process $X(t)$. The ***power spectral density*** (PSD) is defined as

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[|\mathcal{F}\{X_T\}(f)|^2\right] = \lim_{T\to\infty} \frac{1}{2T} \mathbb{E}\left[\left|\int_{-T}^{T} X(t) e^{-j2\pi f t} dt\right|^2\right]$$

We refer to $S_X(f)$ as a density function because it can be interpreted as the amount of power in $X(t)$ in the small band of frequencies from f to $f + df$.

7.16. ***Wiener-Khinchine theorem***: the PSD of a WSS random process is the Fourier transform of its autocorrelation function:

$$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

and

$$R_X(\tau) = \int_{-\infty}^{+\infty} S_X(f) e^{j2\pi f\tau} df.$$

One important consequence is

$$R_X(0) = \mathbb{E}\left[X^2(t)\right] = \int_{-\infty}^{+\infty} S_X(f) df.$$

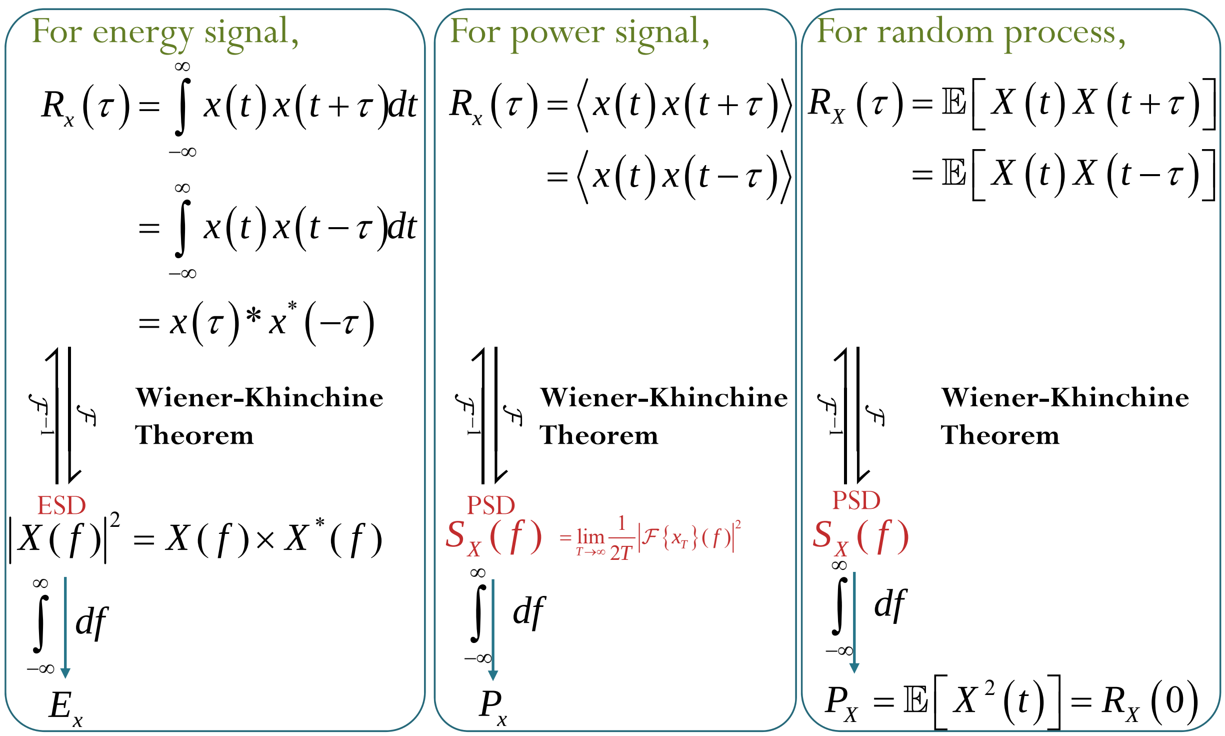


Figure 25: Wiener-Khinchine theorem for different types of signals.

Example 7.17. For the thermal noise in Example 7.14, the corresponding PSD is $S_N(f) = \frac{2kTG}{1+(2\pi f t_0)^2}$ watts/hertz.

7.18. Observe that the thermal noise's PSD in Example 7.17 is approximately flat over the frequency range 0–10 gigahertz. As far as a typical communication system is concerned we might as well let the spectrum be flat over all frequency, i.e.,

$$S_N(f) = \frac{N_0}{2} \text{ watts/hertz,}$$

where N_0 is a constant; in this case $N_0 = 4kTG$.

Definition 7.19. Noise that has a uniform spectrum over the entire frequency range is referred to as **white noise**. In particular, for white noise,

$$S_N(f) = \frac{N_0}{2} \text{ watts/hertz,}$$

- The factor 2 in the denominator is included to indicate that $S_N(f)$ is a two-sided spectrum.
- The adjective "white" comes from white light, which contains equal amounts of all frequencies within the visible band of electromagnetic radiation.
- The average power of white noise is obviously infinite.
  (a) White noise is therefore an abstraction since no physical noise process can truly be white.
  (b) Nonetheless, it is a useful abstraction.
    ◦ The noise encountered in many real systems can be assumed to be approximately white.
      ∗ This is because we can only observe such noise after it has passed through a real system, which will have a finite bandwidth. Thus, as long as the bandwidth of the noise is significantly larger than that of the system, the noise can be considered to have an infinite bandwidth.
    ◦ As a rule of thumb, noise is well modeled as white when its PSD is flat over a frequency band that is 35 times that of the communication system under consideration. [11, p 105]

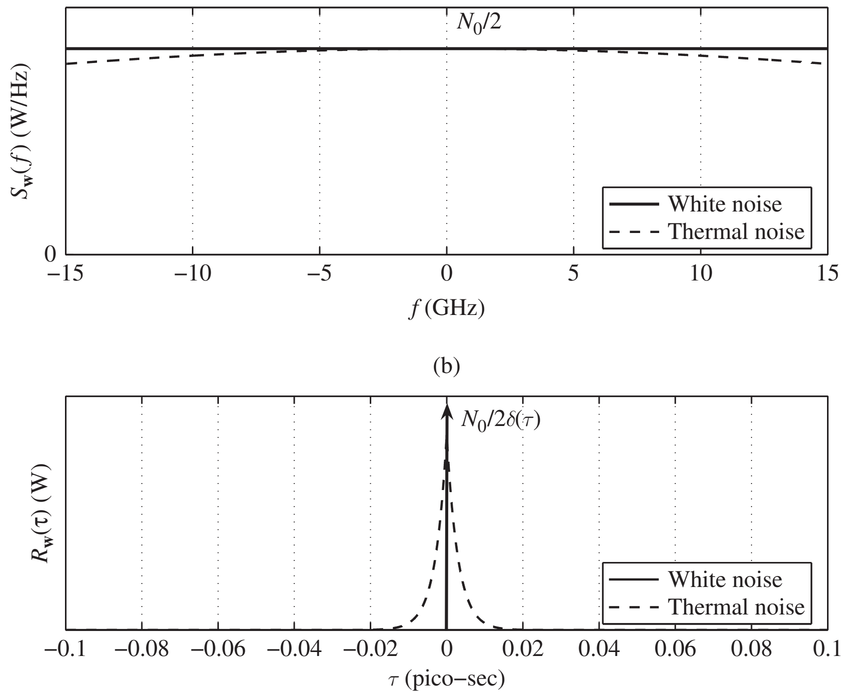


Figure 26: (a) The PSD ($S_N(f)$), and (b) the autocorrelation ($R_N(\tau)$) of noise. (Assume $G = 1/10$ (mhos), $T = 298.15$ K, and $t_0 = 3 \times 10^{-12}$ seconds.) [11, Fig. 3.11]

Theorem 7.20. When we input $X(t)$ through an LTI system whose frequency response is $H(f)$. Then, the PSD of the output $Y(t)$ will be given by

$$S_Y(f) = S_X(f)|H(f)|^2.$$

7.2 Equivalent Vector Channel

7.21. Recall that we are considering the digital modulator/demodulator part shown in Figure 27.

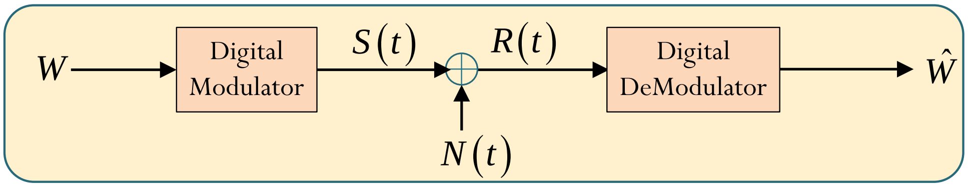


Figure 27: Digital modulator/demodulator and the waveform channel

7.22. The input of the modulator is the (random) message (index) $W \in \{1, 2, \dots M\}$.

- Prior probabilities: $p_j = P[W = j]$.
- Each message is mapped to a waveform to be transmitted over the waveform channel as the transmitted waveform $S(t)$.
  - There are M possible messages. So, there are M waveforms:
  $$s_1(t), s_2(t), \dots, s_M(t).$$
  The (symbol) energy of the j-th waveform is $E_j = \langle s_j(t), s_j(t) \rangle$. The average energy per symbol is $E_s = \sum_{j=1}^{M} p_j E_j$.
  - Transmission of the message $W = j$ is done by inputting the corresponding waveform $s_j(t)$ into the channel.
  Therefore, the probability that the waveform $s_j(t)$ is selected to be transmitted is the same as the probability that the j^{th} message occurs:
  $$p_j = P[W = j] = P[S(t) = s_j(t)]$$

7.23. The noise $N(t)$ in the channel is assumed to be additive. So, the receiver observes $R(t) = S(t) + N(t)$. The noise is also assumed to be independent from the transmitted waveform $S(t)$.

7.24. Conversion of Waveform Channels to Vector Channels:

(a) Given M waveforms $s_1(t), s_2(t), \ldots, s_M(t)$, first find (possibly by GSOP) the K orthonormal basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_K(t)$ for the space spanned by $s_1(t), s_2(t), \ldots, s_M(t)$.

(b) The basis gives the vector representations $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \ldots, \mathbf{s}^{(M)}$ for the waveforms $s_1(t), s_2(t), \ldots, s_M(t)$, respectively. Note that $s_i^{(j)}$, the i^{th} component of the vector $\mathbf{s}^{(j)}$, comes from the inner-product:

$$s_i^{(j)} = \langle s_j(t), \phi_i(t) \rangle .$$

(c) The vector representations of the received waveform and the noise can then be calculated in a similar manner based on the derived basis.

(d) In summary, we convert the waveforms $S(t)$, $R(t)$, and $N(t)$ to their corresponding vectors $\mathbf{S}$, $\mathbf{R}$, and $\mathbf{N}$ by performing inner-product with the orthonormal basis functions: the i-th component of the vector is the inner-product between the waveform and $\phi_i(t)$. In particular,

$$S_i = \langle S(t), \phi_i(t) \rangle, \quad R_i = \langle r(t), \phi_i(t) \rangle, \quad N_i = \langle N(t), \phi_i(t) \rangle .$$

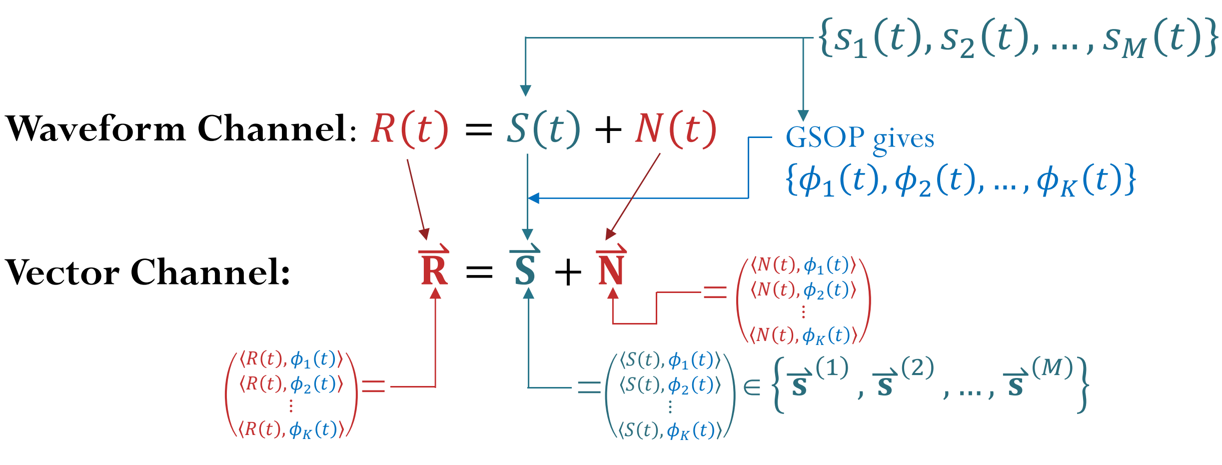


Figure 28: Conversion of Waveform Channels to Vector Channels

Remarks:

- We use the letter K instead of the letter N to represent the number of orthonormal basis functions to avoid the confusion with the random noise which is also denoted by the letter N.

- This conversion is the same as what we did when we convert waveforms to vectors via the GSOP. (See Eq. (35) and Figure 18a.) When $s_j(t)$ is transmitted, the corresponding "transmitted" vector will be $\mathbf{s}^{(j)}$.

Example 7.25. Figure 29 illustrates how the message vectors in quaternary QAM are corrupted by additive noise.

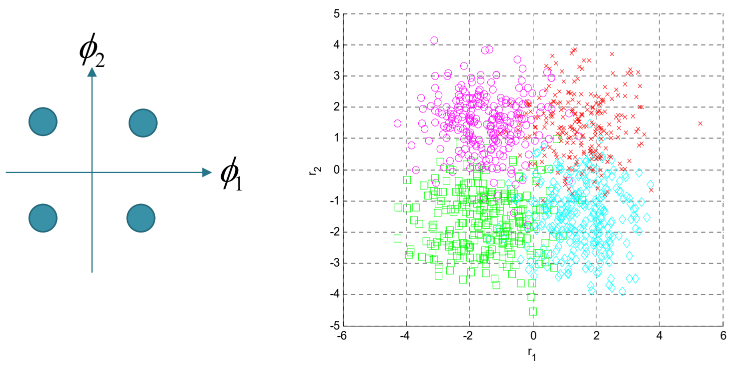


Figure 29: Quaternary QAM: constellation and samples of the received vectors (which are corrupted by additive noise)

7.26. Some facts that followed from the conversion:

(a) For $R(t) = S(t) + N(t)$, we have $\mathbf{R} = \mathbf{S} + \mathbf{N}$.

(b) From the perspective of designing optimal demodulator, the waveform channel and the vector channel are "equivalent".

(c) $E_j = \langle s_j(t), s_j(t) \rangle = \left\langle \mathbf{s}^{(j)}, \mathbf{s}^{(j)} \right\rangle$.

(d) Prior probabilities:

$$p_j = P[W = j] = P[S(t) = s_j(t)] = P\left[\mathbf{S} = \mathbf{s}^{(j)}\right]$$

(e) $\mathbf{S} \perp\!\!\!\perp \mathbf{N}$

(f) When $N(t)$ is a white noise process with $S_N(f) \equiv \frac{N_0}{2}$ (across all frequencies under consideration, we have

(i) $\mathbb{E}[N_j] = 0$, and

(ii) $\mathbb{E}[N_i N_j] = \begin{cases} N_0/2, & i = j, \\ 0, & i \neq j. \end{cases}$

In other words, the noise components are uncorrelated and

$$\mathbb{E}\left[N_i^2\right] = \operatorname{Var} N_i = \frac{N_0}{2}.$$

Modulator and Waveform Channel

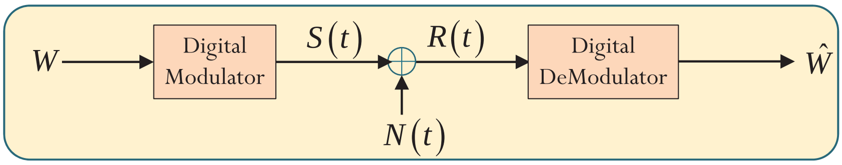
Goal: Want to transmit the message (index) $W \in \{1, 2, 3, \ldots, M\}$

Prior Probabilities: $p_j = P[W = j]$

***M*-ary** Scheme

M =2: Binary
M = 3: Ternary
M = 4: Quaternary

Waveform Channel:



M possible messages requires M possibilities for $S(t)$:

$$\{s_1(t), s_2(t), \ldots, s_M(t)\}$$

$$R(t) = S(t) + N(t)$$

Additive White Noise (Independent of $S(t)$)

Transmitted waveform

Received waveform

Transmission of the message $W = j$ is done by inputting the corresponding waveform $s_j(t)$ into the channel.

Prior Probabilities: $p_j = P[W = j] = P[S(t) = s_j(t)]$

Energy: $E_j = \langle s_j(t), s_j(t) \rangle$ $\quad E_s = \sum_{j=1}^{M} p_j E_j = (\log_2 M) E_b$

Conversion to Vector Channels

Waveform Channel: $R(t) = S(t) + N(t)$

Vector Channel $\vec{\mathbf{R}} = \vec{\mathbf{S}} + \vec{\mathbf{N}}$

Use GSOP to find K orthonormal basis functions $\{\phi_1(t), \phi_2(t), \ldots, \phi_K(t)\}$ for the space spanned by $\{s_1(t), s_2(t), \ldots, s_M(t)\}$.

This gives vector representations for the waveforms $s_1(t), s_2(t), \ldots, s_M(t)$:

$$\vec{\mathbf{s}}^{(1)}, \vec{\mathbf{s}}^{(2)}, \ldots, \vec{\mathbf{s}}^{(M)}$$

which can be visualized in the form of signal constellation

Note that $S_i^{(j)}$, the i^{th} component of the $\vec{\mathbf{S}}$ vector, comes from the inner-product:

$$S_i^{(j)} = \langle S(t), \phi_i(t) \rangle$$

The received vector $\vec{R}$ is computed in the same way: the j component is given by

$$R_i = \langle r(t), \phi_i(t) \rangle$$

In which case, the corresponding noise vector $\vec{\mathbf{N}}$ is computed in the same way: the j component is given by

$$N_i = \langle N(t), \phi_i(t) \rangle$$

Prior Probabilities:

$$p_j = P[W = j] = P[S(t) = s_j(t)]$$
$$= P\left[\vec{\mathbf{S}} = \vec{\mathbf{s}}^{(j)}\right]$$

For additive white **Gaussian** noise (AWGN) process $N(t)$,

$$N_i \sim N \sim \mathcal{N}\left(0, \frac{N_0}{2}\right) = \mathcal{N}(0, \sigma^2) \Rightarrow \vec{\mathbf{N}} \sim \mathcal{N}\left(\vec{\mathbf{0}}, \frac{N_0}{2} I\right) \Rightarrow f_{\vec{\mathbf{N}}}(\vec{\mathbf{n}}) = \frac{1}{(2\pi)^{\frac{K}{2}} \sigma^K} e^{-\frac{1}{2}\frac{\|\vec{\mathbf{n}}\|^2}{\sigma^2}}$$