Digital Communication Systems
ECS 452

Asst. Prof. Dr. Prapun Suksompong

prapun@siit.tu.ac.th

7. The Waveform Channel

Office Hours:

BKD, 4th floor of Sirindhralai building

Monday	14:00-16:00
Thursday	10:30-11:30
Friday	12:00-13:00

The Big Plan

- Although we are thinking about digital communication systems,
- actual signaling in the wire or air is in continuous time which is described by the **waveform channel**:
$$R(t) = S(t) + N(t).$$
- Directly finding the optimal (MAP) detector or evaluating the performance $P(\mathcal{E})$ of such system is difficult.
- Our approach is to first construct an equivalent **vector channel** that preserves the relevant features. (Chapter 7)
- Then, at the end, we can use what we learn to go back to the original waveform channel and directly work with the waveforms.

Review: ECS315

ECS 315: Probability and Random Processes **2016/1**

HW Solution 12 — Due: November 29, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Review: ECS315

Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t + \theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

Review: ECS315

11.5 Linear Dependence

Definition 11.55. Given two random variables X and Y, we may calculate the following quantities:

(a) **Correlation**: $\mathbb{E}[XY]$. $= \mathbb{E}[YX]$

(b) **Covariance**: $\operatorname{Cov}[X,Y] = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]$. $= \operatorname{Cov}[Y,X]$

(c) **Correlation coefficient**: $\rho_{X,Y} = \frac{\operatorname{Cov}[X,Y]}{\sigma_X \sigma_Y}$

standardize the RVs first

Exercise 11.56 (F2011). Continue from Exercise 11.9.

(a) Find $\mathbb{E}[XY]$.

(b) Check that $\operatorname{Cov}[X,Y] = -\frac{1}{25}$.

11.57. $\operatorname{Cov}[X,Y] = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)] = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y$

m_X m_Y

[ECS315 2016]

(Cross-)Correlation Functions

For energy signals,

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt = \langle x(t), y(t+\tau) \rangle$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\,y(t)\,dt = \langle x(t-\tau), y(t) \rangle$$

inner-product operator

For power signals,

$$R_{x,y}(\tau) = \langle x(t)\,y(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,y(t+\tau)\,dt$$

$$= \langle x(t-\tau)\,y(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-\tau)\,y(t)\,dt$$

time-average operator

For periodic signals,

$$R_{x,y}(\tau) = \frac{1}{T_0} \int_{T_0} x(t)\,y(t+\tau)\,dt$$

common period

For random variables,

$$\mathbb{E}[XY]$$

For random processes,

$$R_{X,Y}(t_1,t_2) = \mathbb{E}[X(t_1)\,Y(t_2)]$$

expectation operator

For WSS random processes,

$$R_{X,Y}(\tau) = \mathbb{E}[X(t)\,Y(t+\tau)]$$

$$= \mathbb{E}[X(t-\tau)\,Y(t)]$$

(Auto-)Correlation Functions

For an energy signal,

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt = \langle x(t), x(t+\tau) \rangle$$

$$= \int_{-\infty}^{\infty} x(t-\tau)x(t)dt = \langle x(t-\tau), x(t) \rangle$$

inner-product operator

For a random process,

$$R_X(t_1, t_2) = \mathbb{E}\left[X(t_1) X(t_2) \right]$$

expectation operator

For a power signal,

$$R_x(\tau) = \langle x(t)x(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

$$= \langle x(t-\tau)x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t-\tau)x(t)dt$$

time-average operator

For a periodic signal,

$$R_x(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t+\tau)dt$$

common period

For a WSS random process,

$$R_X(\tau) = \mathbb{E}\left[X(t) X(t+\tau) \right]$$

$$= \mathbb{E}\left[X(t-\tau) X(t) \right]$$

Review: ECS315

ECS 315: Probability and Random Processes **2016/1**

HW Solution 12 — Due: November 29, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

Solution:

(b) Y is another function of Θ.

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[5\cos(7t_1+\Theta) \times 5\cos(7t_2+\Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1+\theta) \times 5\cos(7t_2+\theta) d\theta \\ &= \frac{25}{2\pi} \int_0^{2\pi} \cos(7t_1+\theta) \times \cos(7t_2+\theta) d\theta.\end{aligned}$$

Recall[1] the cosine identity

$$\cos(a) \times \cos(b) = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}Y &= \frac{25}{4\pi} \int_0^{2\pi} \cos(14t + 2\theta) + \cos(7(t_1 - t_2))\, d\theta \\ &= \frac{25}{4\pi}\left(\int_0^{2\pi} \cos(14t + 2\theta)\, d\theta + \int_0^{2\pi} \cos(7(t_1-t_2))\, d\theta\right).\end{aligned}$$

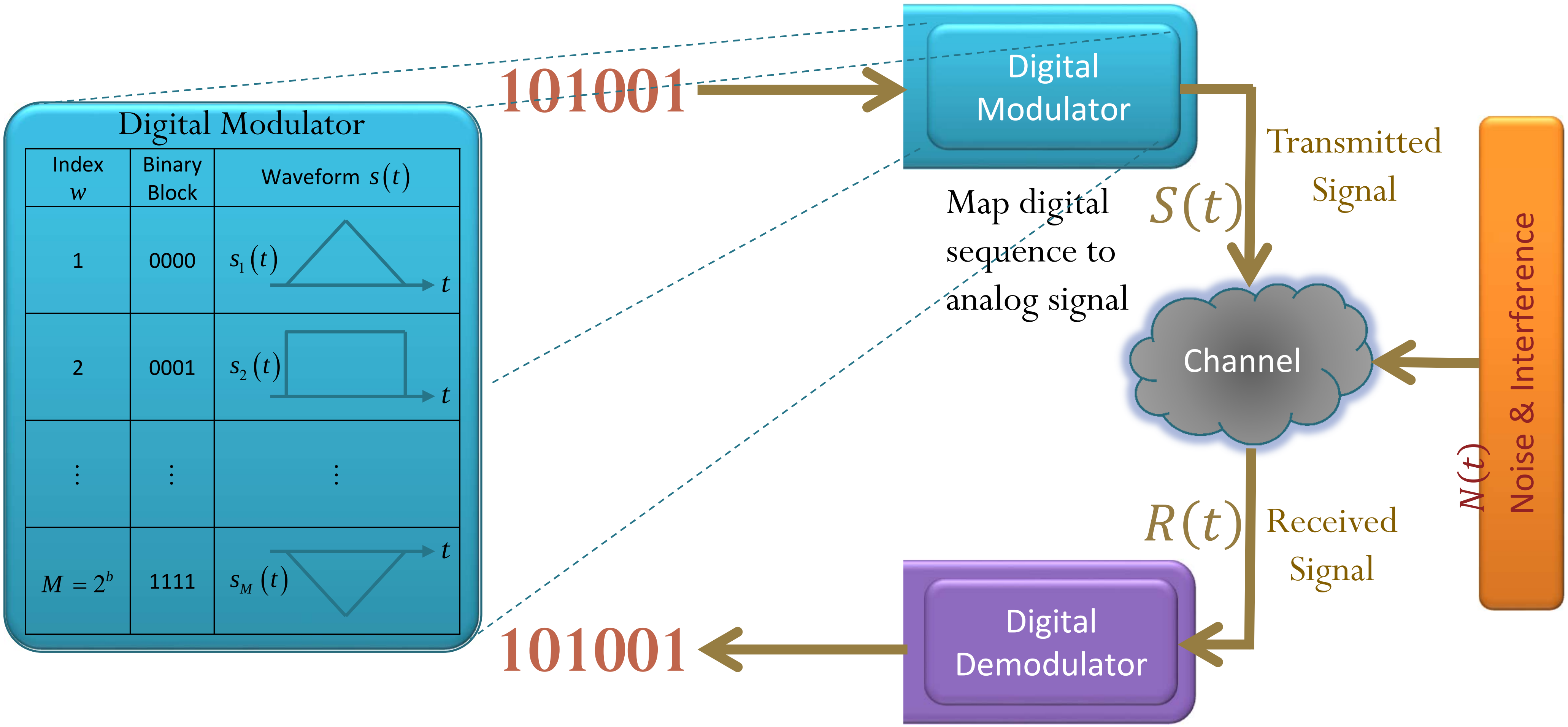
The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1-t_2)) \int_0^{2\pi} d\theta = \frac{25}{4\pi}\cos(7(t_1-t_2))\, 2\pi = \boxed{\frac{25}{2}\cos(7(t_1-t_2))}.$$

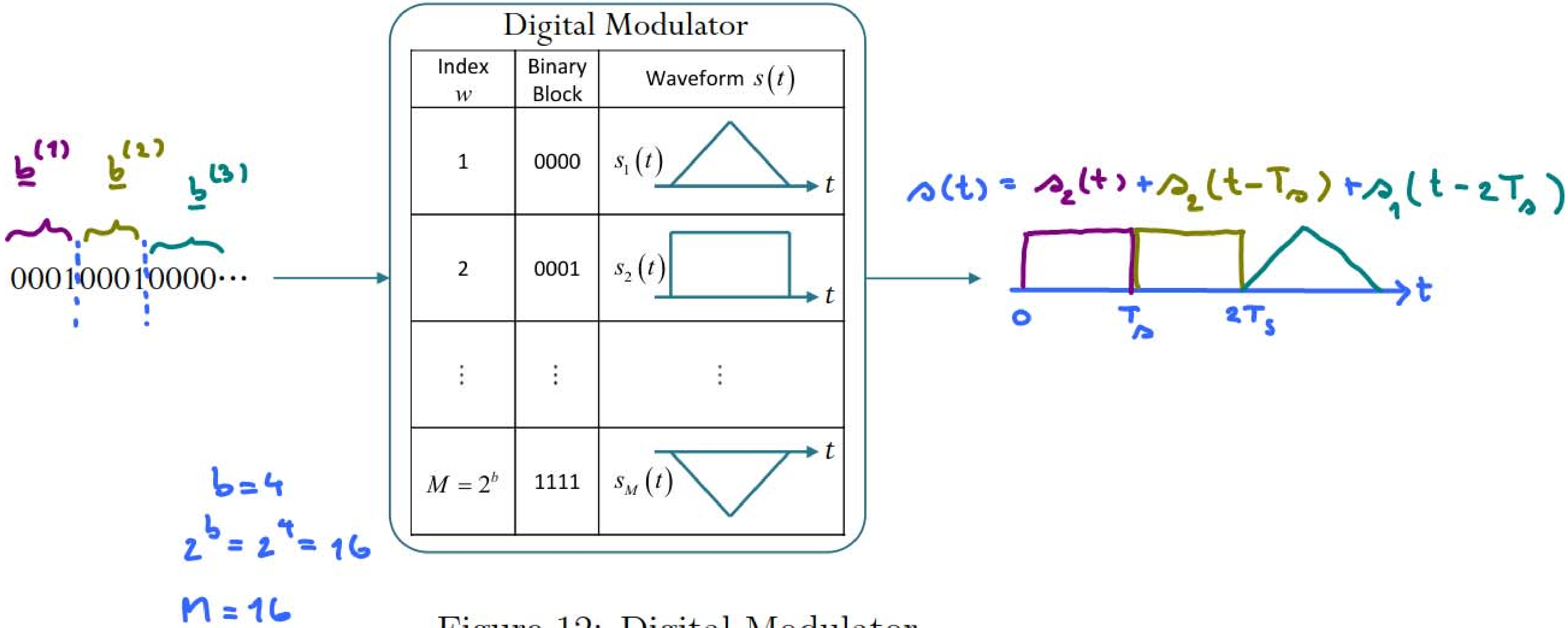
[1] This identity could be derived easily via the Euler's identity:

$$\begin{aligned}\cos(a) \times \cos(b) &= \frac{e^{ja}+e^{-ja}}{2} \times \frac{e^{jb}+e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb} + e^{-ja}e^{jb} + e^{ja}e^{-jb} + e^{-ja}e^{-jb}\right) \\ &= \frac{1}{2}\left(\frac{e^{ja}e^{jb} + e^{-ja}e^{-jb}}{2} + \frac{e^{-ja}e^{jb} + e^{ja}e^{-jb}}{2}\right) \\ &= \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).\end{aligned}$$

Digital Modulation/Demodulation



Review: Digital Modulator



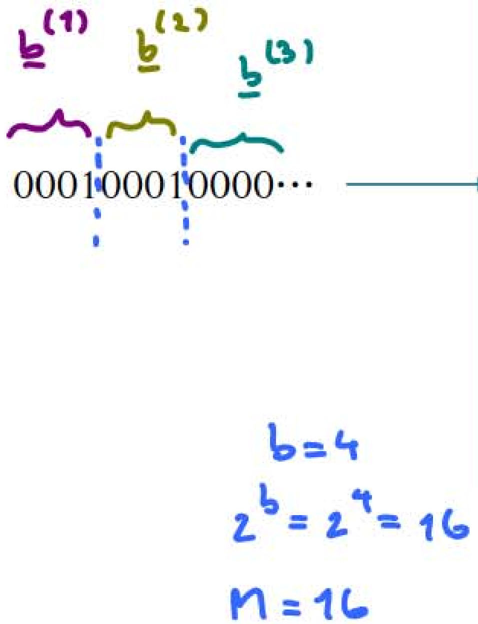
Digital Modulator

Index w	Binary Block	Waveform $s(t)$
1	0000	$s_1(t)$
2	0001	$s_2(t)$
⋮	⋮	⋮
$M = 2^b$	1111	$s_M(t)$

Figure 12: Digital Modulator

[Lecture Notes: Chapter 6 p. 67]

Review: Digital Modulator



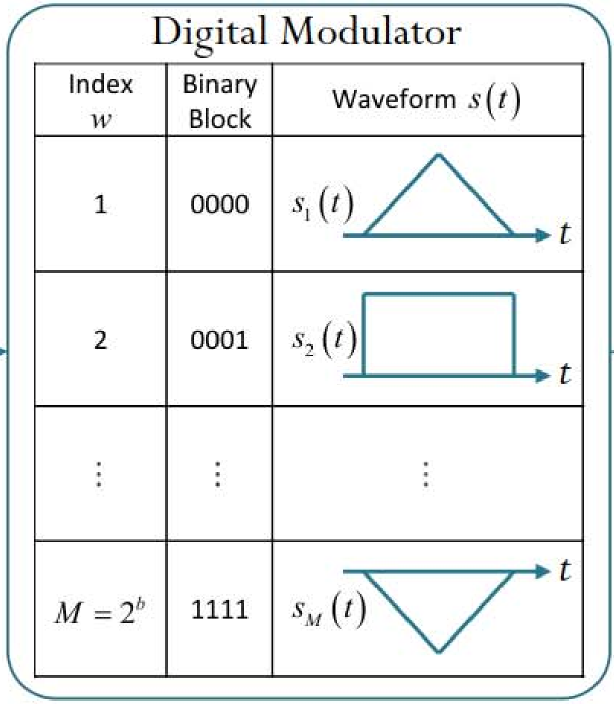
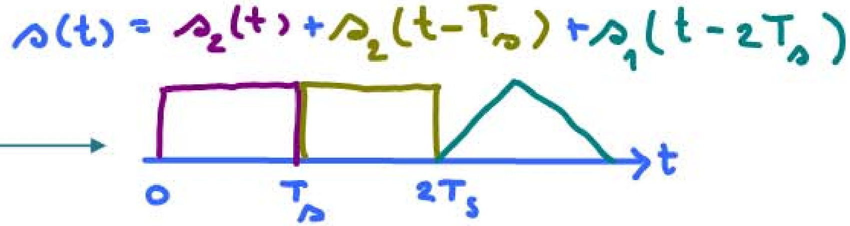


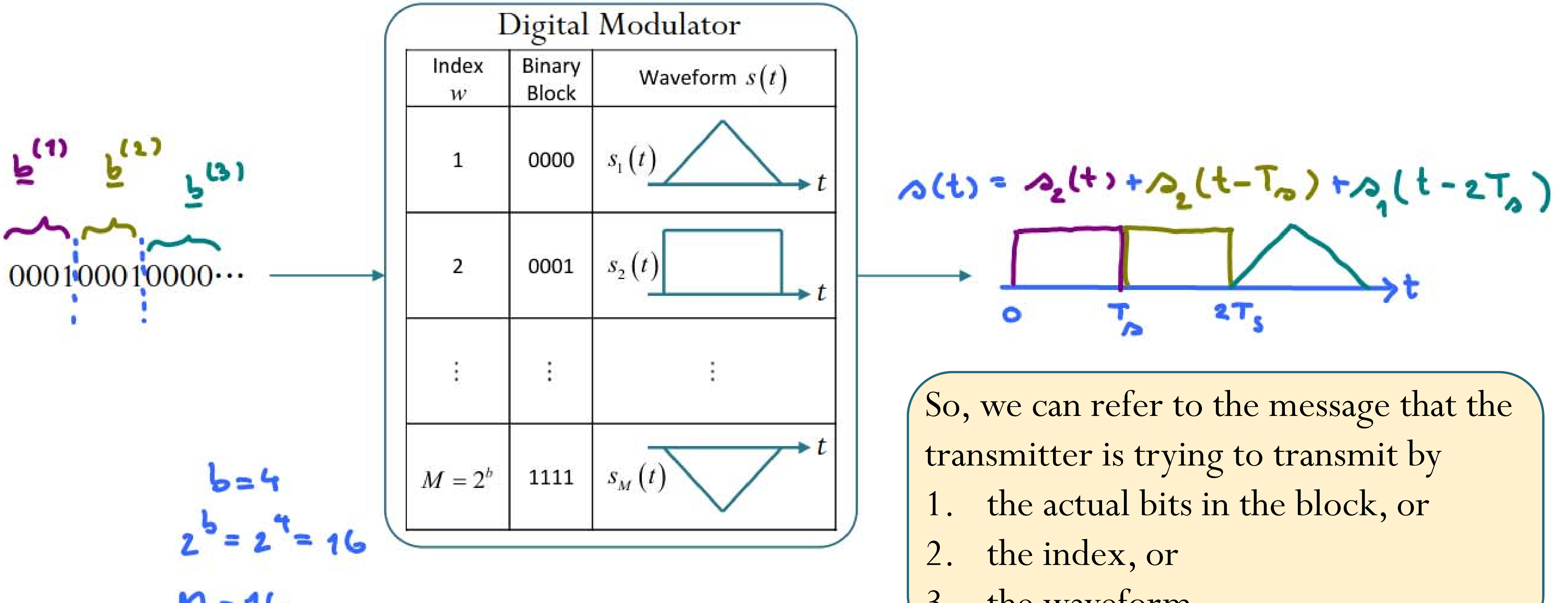
Figure 12: Digital Modulator



The following statements are equivalent:

1. The transmitter is trying to transmit a 4-bit message 0001.
2. The transmitter is trying to transmit message #2.
3. The transmitter is trying to transmit the waveform $s_1(t)$.

Review: Digital Modulator



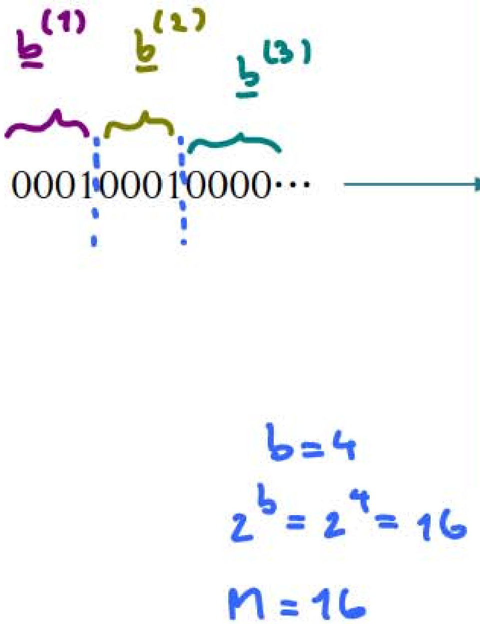
Index w	Binary Block	Waveform $s(t)$
1	0000	$s_1(t)$
2	0001	$s_2(t)$
⋮	⋮	⋮
$M = 2^b$	1111	$s_M(t)$

Figure 12: Digital Modulator

So, we can refer to the message that the transmitter is trying to transmit by

1. the actual bits in the block, or
2. the index, or
3. the waveform.

Review: Digital Modulator



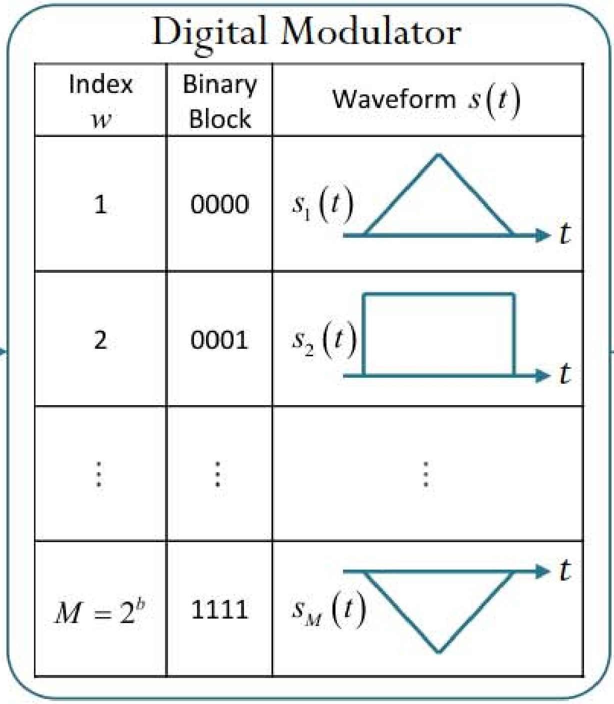
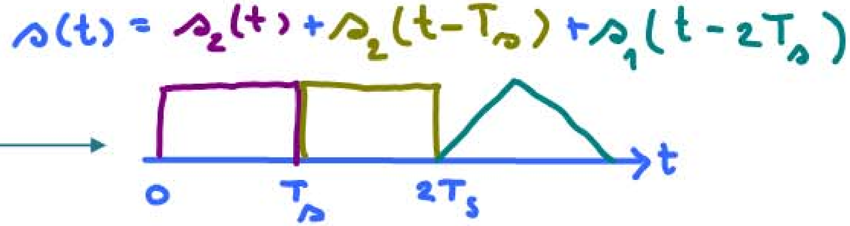


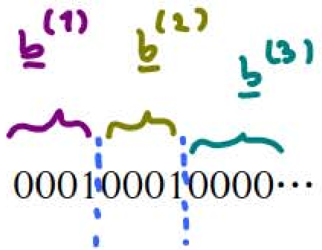
Figure 12: Digital Modulator



The following statements are equivalent:

1. The transmitter is trying to transmit a 4-bit message 0001.
2. The transmitter is trying to transmit message #2.
3. The transmitter is trying to transmit the waveform $s_1(t)$.
4. The transmitter is trying to transmit the message vector $\vec{\mathbf{s}}^{(1)}$

Review: Digital Modulator



Digital Modulator

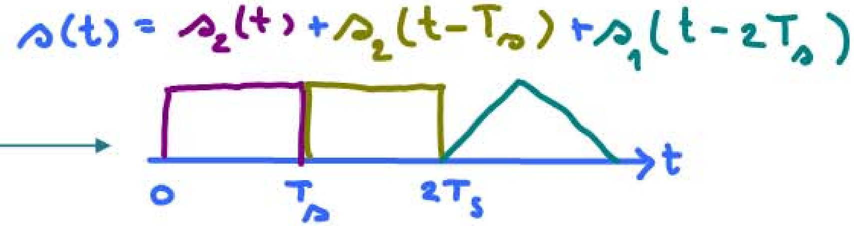
Index w	Binary Block	Waveform $s(t)$
1	0000	$s_1(t)$
2	0001	$s_2(t)$
⋮	⋮	⋮
$M = 2^b$	1111	$s_M(t)$

$b = 4$

$2^b = 2^4 = 16$

$M = 16$

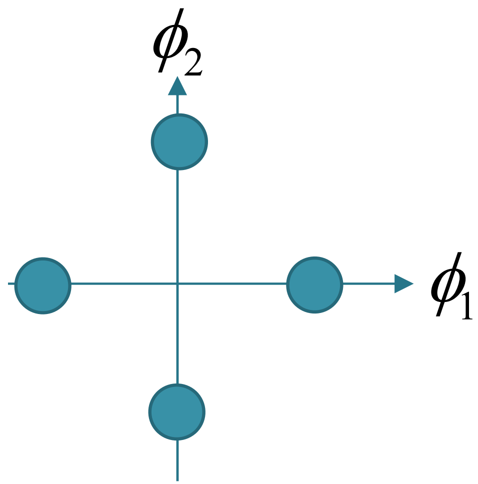
Figure 12: Digital Modulator

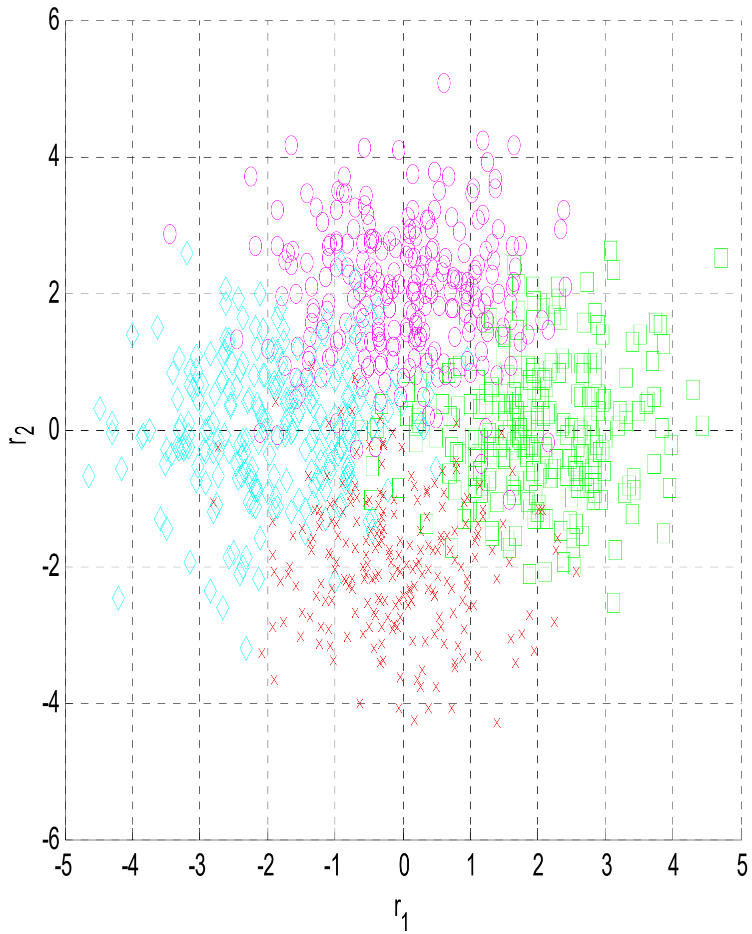


So, we can refer to the message that the transmitter is trying to transmit by

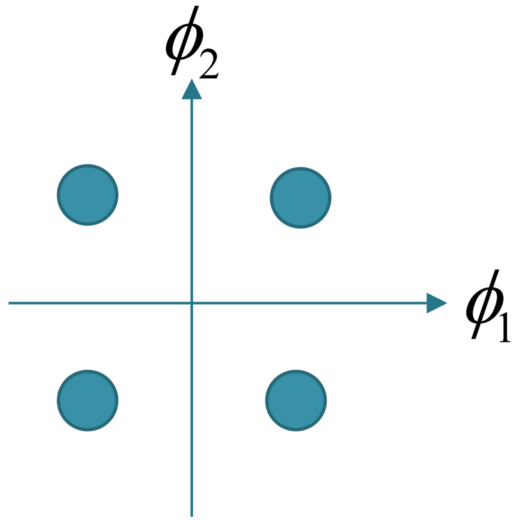
1. the actual bits in the block, or
2. the index, or
3. the waveform, or
4. The vector representation

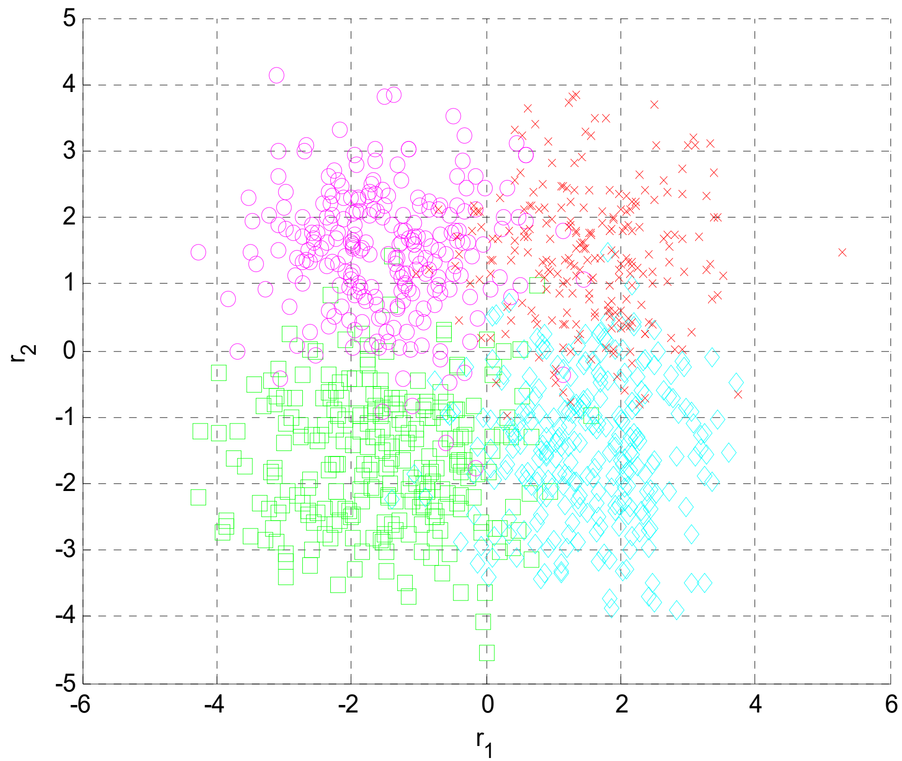
Standard Quaternary PSK





Standard Quaternary QAM





Modulator and Waveform Channel

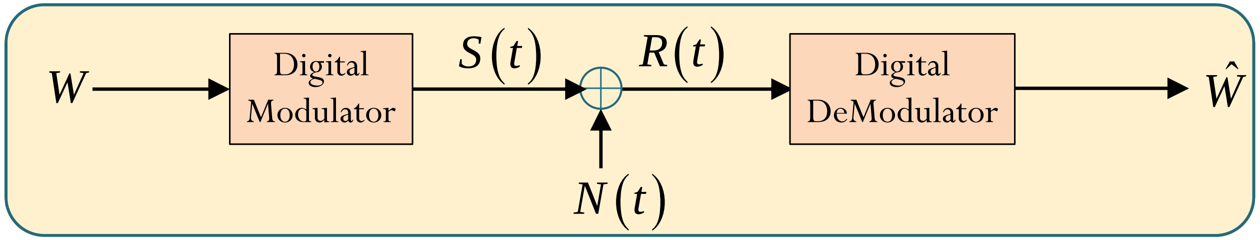
Goal: Want to transmit the message (index) $W \in \{1, 2, 3, \ldots, M\}$

Prior Probabilities: $p_j = P[W = j]$

***M*-ary** Scheme

M = 2: Binary
M = 3: Ternary
M = 4: Quaternary

Waveform Channel:



M possible messages requires M possibilities for $S(t)$:

$$\{s_1(t), s_2(t), \ldots, s_M(t)\}$$

$$R(t) = S(t) + N(t)$$

Additive White Noise (Independent of $S(t)$)

Transmitted waveform

Received waveform

Transmission of the message $W = j$ is done by inputting the corresponding waveform $s_j(t)$ into the channel.

Prior Probabilities: $p_j = P[W = j] = P[S(t) = s_j(t)]$

Energy: $E_j = \langle s_j(t), s_j(t) \rangle \qquad E_s = \sum_{j=1}^{M} p_j E_j = (\log_2 M) E_b$

Conversion to Vector Channels

Waveform Channel: $R(t) = S(t) + N(t)$

Vector Channel $\vec{\mathbf{R}} = \vec{\mathbf{S}} + \vec{\mathbf{N}}$

Note that $S_i^{(j)}$, the i^{th} component of the $\vec{\mathbf{S}}$ vector, comes from the inner-product:

$$S_i^{(j)} = \langle S(t), \phi_i(t) \rangle$$

The received vector $\vec{R}$ is computed in the same way: the j component is given by

$$R_i = \langle r(t), \phi_i(t) \rangle$$

In which case, the corresponding noise vector $\vec{\mathbf{N}}$ is computed in the same way: the j component is given by

$$N_i = \langle N(t), \phi_i(t) \rangle$$

Use GSOP to find K orthonormal basis functions $\{\phi_1(t), \phi_2(t), \dots, \phi_K(t)\}$ for the space spanned by $\{s_1(t), s_2(t), \dots, s_M(t)\}$.

This gives vector representations for the waveforms $s_1(t), s_2(t), \dots, s_M(t)$:

$$\vec{\mathbf{s}}^{(1)}, \vec{\mathbf{s}}^{(2)}, \dots, \vec{\mathbf{s}}^{(M)}$$

which can be visualized in the form of signal constellation

Prior Probabilities:

$$p_j = P[W = j] = P\left[S(t) = s_j(t)\right]$$

$$= P\left[\vec{\mathbf{S}} = \vec{\mathbf{s}}^{(j)}\right]$$

For additive white **Gaussian** noise (AWGN) process $N(t)$,

$$N_i \sim N \sim \mathcal{N}\left(0, \frac{N_0}{2}\right) = \mathcal{N}\left(0, \sigma^2\right) \Rightarrow \vec{\mathbf{N}} \sim \mathcal{N}\left(\vec{\mathbf{0}}, \frac{N_0}{2} I\right) \Rightarrow f_{\vec{\mathbf{N}}}(\vec{\mathbf{n}}) = \frac{1}{(2\pi)^{\frac{K}{2}} \sigma^K} e^{-\frac{1}{2}\frac{\|\vec{\mathbf{n}}\|^2}{\sigma^2}}$$