ECS 452: Digital Communication Systems **2015/2**

HW 8 — Due: Not Due

Lecturer: Asst. Prof. Dr. Prapun Suksompong

Problem 1. In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-3, 3\}$ with $p_1 = P[S = -3] = 0.3$ and $p_2 = P[S = 3] = 0.7$. The message is corrupted by an independent additive noise N which is uniform on $[-4, 4]$.

(a) Find the pdf of the noise.

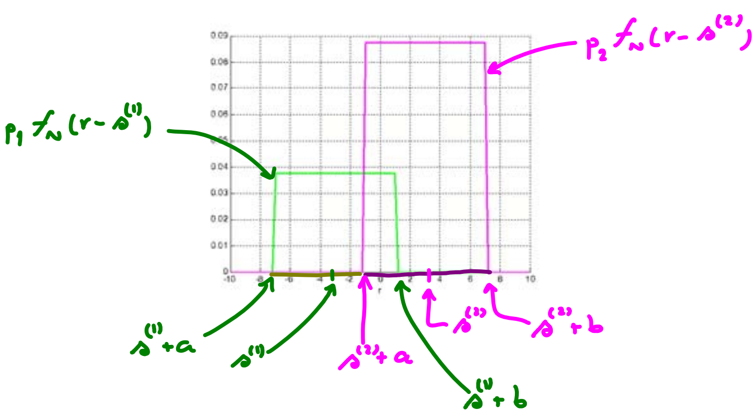
Recall that the uniform pdf on $[a,b]$ is given by $f_N(n) = \begin{cases} \frac{1}{b-a}, & a<n<b, \\ 0, & \text{otherwise.} \end{cases}$

Here, $a=4$ and $b=-4$. So, $f_N(n) = \begin{cases} 1/8, & -4<n<4, \\ 0, & \text{otherwise.} \end{cases}$

(b) Find the MAP detector $\hat{s}_{\mathrm{MAP}}(r)$.

The MAP detector is given by $\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r-s)$. This is true regardless of the pdf of the noise.

Here, there are two possible values for s: -3 and 3. So we compare $\underset{0.3}{p_1} f_N(r - \underset{-3}{s^{(1)}})$ and $\underset{0.7}{p_2} f_N(r - \underset{3}{s^{(2)}})$.



Observe that

① When $s^{(1)}+a < r < s^{(2)}+a$,

$p_1 f_N(r-s^{(1)}) > p_2 f_N(r-s^{(2)})$.

Therefore, $\hat{s}_{MAP}(r) = s^{(1)}$ in this region.

② When $s^{(2)}+a < r < s^{(1)}+b$,

$p_1 f_N(r-s^{(1)}) < p_2 f_N(r-s^{(2)})$.

Therefore, $\hat{s}_{MAP}(r) = s^{(2)}$ in this region.

③ When $r < s^{(1)}+a$ or $r > s^{(2)}+b$, the pdf in both cases are 0.

So, these are the impossible regions. The received signal R won't fall in these regions. Therefore, it does not matter how the detector behaves in this region.

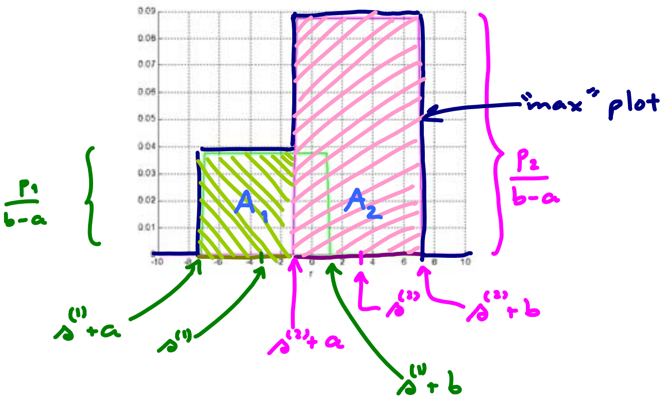
Conslusion: $\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & s^{(1)}+a < r < s^{(2)}+a, \\ s^{(2)}, & s^{(2)}+a < r < s^{(2)}+b, \\ \text{anything}, & \text{otherwise} \end{cases} = \begin{cases} s^{(1)}, & r < s^{(2)}+a \\ s^{(2)}, & r \geq \underbrace{s^{(2)}+a} \end{cases} = \begin{cases} -3, & r < -1 \\ 3, & r \geq -1 \end{cases}$

$\tau^* = s^{(2)}+a = 3+(-4) = -1$

Note that at $r = s^{(1)}+a,\ s^{(2)}+a,\ s^{(2)}+b$ (the boundaries), the comparison of $p_1 f_N(r-s^{(1)})$ and $p_2 f_N(r-s^{(2)})$ does not matter because the probabilities that the reveived signal R will be exactly any one of these three points is zero. Therefore, we can define $\hat{s}_{MAP}(r)$ to be anything here.
To further simplify the expression, we choose the "anything" parts above such that they can be combined into adjacent intervals. This gives

(c) Evaluate the error probability of the MAP detector.

Recall that for MAP detector, $P(\mathcal{C})$ = area under the "max" plot.



Area $A_1 = \frac{p_1}{b-a} \times \left((s^{(2)}+\cancel{a}) - (s^{(1)}+\cancel{a})\right) = \frac{p_1}{b-a}\left(s^{(2)} - s^{(1)}\right)$

Area $A_2 = \frac{p_2}{b-a} \times \left((\cancel{s^{(2)}}+b) - (\cancel{s^{(2)}}+a)\right) = p_2$

So, $P(\mathcal{C}) = \frac{p_1}{b-a}\left(s^{(2)} - s^{(1)}\right) + p_2$.

Therefore, $P(\mathcal{E}) = 1 - P(\mathcal{C}) = p_1 - \frac{p_1}{b-a}\left(s^{(2)} - s^{(1)}\right) = p_1\left(1 - \frac{s^{(2)} - s^{(1)}}{b-a}\right)$

$= 0.3\left(1 + \frac{3-(-3)}{4-(-4)}\right) = 0.3\left(1 - \frac{3}{4}\right) = 0.3 \times \frac{1}{4} = 0.075$

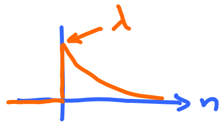
Problem 2. In a binary antipodal signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-3, 3\}$ with $p_1 = P[S = -3] = 0.3$ and $p_2 = P[S = 3] = 0.7$. The message is corrupted by an independent additive exponential noise N whose pdf is

$$f_N(n) = \begin{cases} \frac{1}{2} e^{-n/2}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

Some facts about exponential noise:

(1) $f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$

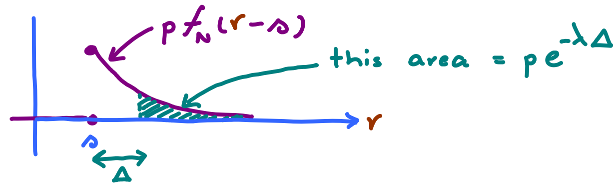


In this question, $\lambda = \frac{1}{2}$ and $\mathbb{E}N = \frac{1}{\lambda} = 2$.

(2) $\mathbb{E}N = \frac{1}{\lambda}$ and $\operatorname{Var} N = \frac{1}{\lambda^2}$

(3) MATLAB use $\mathbb{E}N$ as the parameter instead of λ

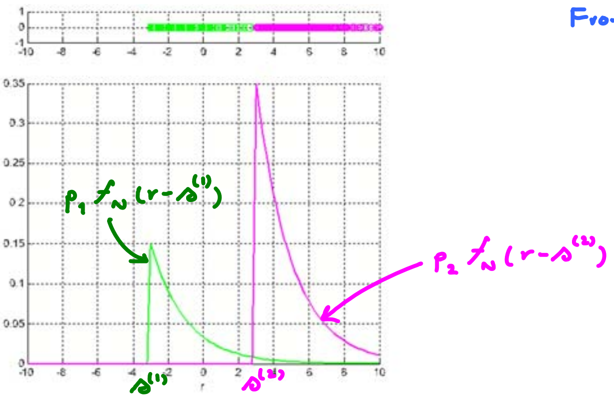
(4) $P[N > n] = \int_n^\infty f_N(n)\,dn = \begin{cases} e^{-\lambda n}, & n \geq 0 \\ 1, & n < 0 \end{cases}$ $\Rightarrow$



The MAP detector is given by $\hat{s}_{MAP}(r) = \underset{s}{\arg\max}\; p_s f_N(r-s)$.

This is true regardless of the pdf of the noise.

Here, there are two possible values for s: -3 and 3. So we compare $\underset{0.3}{p_1} f_N(r-\underset{-3}{s^{(1)}})$ and $\underset{0.7}{p_2} f_N(r-\underset{3}{s^{(2)}})$.



From the graph, it is clear that

1. $R < s^{(1)}$ is impossible.

   So, the detector can do anything in this region without affecting its performance.

2. When $s^{(1)} < r < s^{(2)}$,

   $$p_1 f_N(r-s^{(1)}) > p_2 f_N(r-s^{(2)}).$$

   So in this region, $\hat{s}_{MAP}(r) = s^{(1)}$.

3. When $r > s^{(2)}$,

   $$p_1 f_N(r-s^{(1)}) < p_2 f_N(r-s^{(2)}).$$

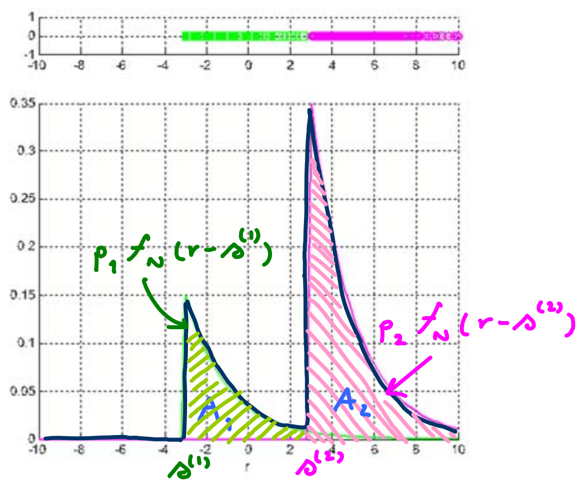
   So, in this region, $\hat{s}_{MAP}(r) = s^{(2)}$.

Conclusion:

$$\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & s^{(1)} < r < s^{(2)}, \\ s^{(2)}, & r > s^{(2)}, \\ \text{anything}, & \text{otherwise} \end{cases} \underset{\text{simplification}}{=} \begin{cases} s^{(1)}, & r < s^{(2)} \\ s^{(2)}, & r \geq s^{(2)} \end{cases} = \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

$\tau^* = s^{(2)} = 3$

(b) Evaluate the error probability of the MAP detector.



Recall that for MAP detector,

$P(\mathcal{C})$ = area under the "max" plot.

Area $A_1 = p_1 - p_1 e^{-\lambda(s^{(2)} - s^{(1)})}$

Area $A_2 = p_2$

So, $P(\mathcal{C}) = 1 - p_1 e^{-\lambda(s^{(2)} - s^{(1)})}$ and

$$P(\mathcal{E}) = 1 - P(\mathcal{C}) = p_1 e^{-\lambda(s^{(2)} - s^{(1)})}$$

$$= 0.3\, e^{-\frac{1}{2}(3-(-3))} = 0.3\, e^{-3} \approx 0.0149$$

Three-point constellation (M = 3)

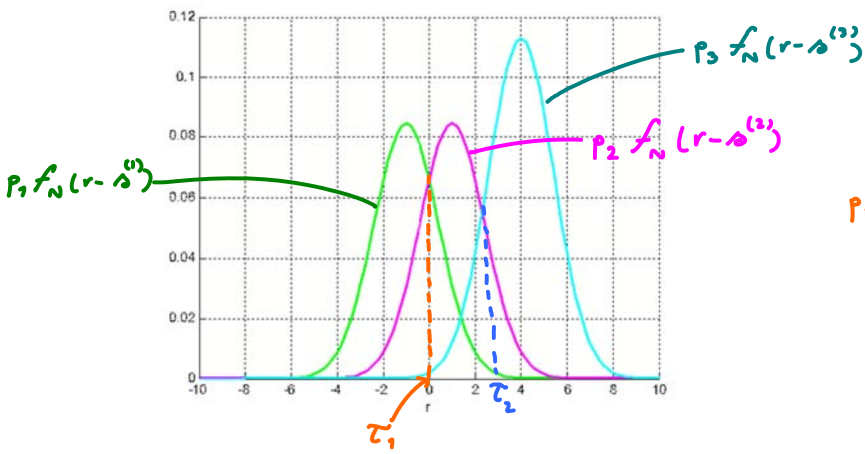
Problem 3. In a ternary signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-1, 1, 4\}$ with $p_1 = P[S = -1] = 0.3 = p_2 = P[S = 1]$ and $p_3 = P[S = 4] = 0.4$. The message is corrupted by an independent additive Gaussian noise $N \sim \mathcal{N}(0, 2)$.

(a) Find the average signal energy[1] E_s.

E_s = Average energy per symbol $= (-1)^2 \times 0.3 + 1^2 \times 0.3 + 4^2 \times 0.4 = 0.3 + 0.3 + 6.4 = 7$

(b) Find the MAP detector $\hat{s}_{\text{MAP}}(r)$.

To find the MAP detector, we compare $p_1 f_N(r - s^{(1)})$, $p_2 f_N(r - s^{(2)})$, and $p_3 f_N(r - s^{(3)})$



To find τ_1, we find r such that

$$p_1 f_N(r - s^{(1)}) = p_2 f_N(r - s^{(2)}).$$

$$p_1 \cancel{\frac{1}{\sqrt{2\pi}\sigma}} e^{-\frac{1}{2}\left(\frac{r - s^{(1)}}{\sigma}\right)^2} = p_2 \cancel{\frac{1}{\sqrt{2\pi}\sigma}} e^{-\frac{1}{2}\left(\frac{r - s^{(2)}}{\sigma}\right)^2}$$

$$r = \boxed{\frac{\sigma^2}{s^{(2)} - s^{(1)}} \ln \frac{p_1}{p_2} + \frac{s^{(1)} + s^{(2)}}{2}}$$

This is the same formula that we derived in the lecture note to find τ_{MAP} when M = 2

Here, $p_1 = p_2$. So, $\tau_1 = \frac{s^{(1)} + s^{(2)}}{2} = \frac{-1+1}{2} = 0$.

Similarly, we can find τ_2 by $\frac{\sigma^2}{s^{(3)} - s^{(2)}} \ln \frac{p_3}{p_2} + \frac{s^{(2)} + s^{(3)}}{2} = 2.3082$

The MAP detector is given by

$$\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & r \le \tau_1 \\ s^{(2)}, & \tau_1 < r \le \tau_2 \\ s^{(3)}, & r > \tau_2 \end{cases} = \begin{cases} -1, & r \le 0, \\ 1, & 0 < r \le 2.3082, \\ 4, & r > 2.3082. \end{cases}$$

[1] Same as "average symbol energy" or "average energy per symbol" or "average energy per signal"

(c) Indicate the decision regions of the MAP detector in part (b).

$\mathcal{D}_1 = \{ r : \hat{s}_{MAP}(r) = s^{(1)} \} = (-\infty, 0]$

$\mathcal{D}_2 = \{ r : \hat{s}_{MAP}(r) = s^{(2)} \} = (0, 2.3082]$

$\mathcal{D}_3 = \{ r : \hat{s}_{MAP}(r) = s^{(3)} \} = (2.3082, \infty)$

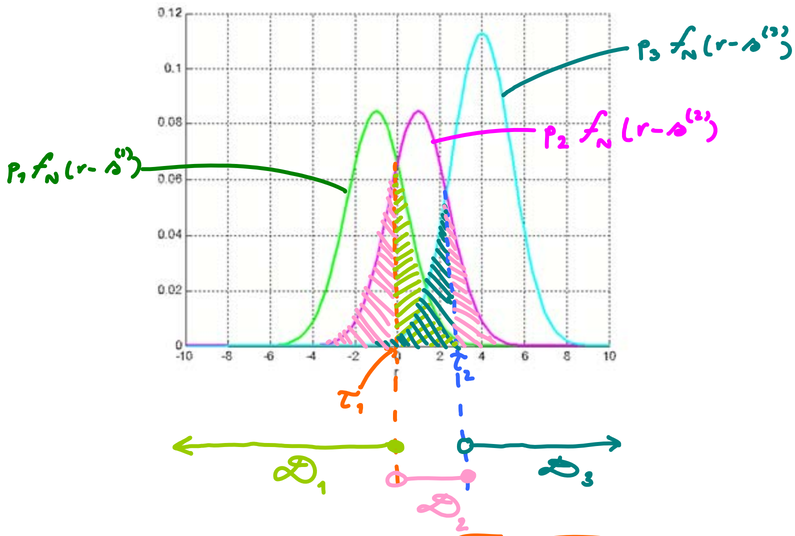
(d) Evaluate the error probability of the MAP detector.

$$P(\mathcal{E}) = \sum_{j=1}^{M} \int_{\mathcal{D}_j} p_j f_N(r - s^{(j)})\, dr$$

$$= p_1 Q\left(\frac{\tau_1 - s^{(1)}}{\sigma}\right) + \left(Q\left(\frac{s^{(2)} - \tau_1}{\sigma}\right) + Q\left(\frac{\tau_2 - s^{(2)}}{\sigma}\right)\right) p_2$$

$$+ \; p_3 Q\left(\frac{s^{(3)} - \tau_2}{\sigma}\right)$$

$= 0.2434$



Problem 4. In a **standard** quaternary signaling scheme, the message S is equiprobably selected from the alphabet set $\mathcal{S} = \left\{-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}\right\}$. The message is corrupted by an independent additive exponential noise N whose pdf is

$p_j = \frac{1}{4}$.

$$f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the average symbol energy.

$$E_s = \sum_{j=1}^{M} p_j |s^{(j)}|^2 = \frac{1}{4}\left(\left(-\frac{3}{2}d\right)^2 + \left(-\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 + \left(\frac{3}{2}d\right)^2\right) = \frac{1}{16} d^2 (9+1+1+9) = \frac{20}{16} d^2 = \frac{5}{4} d^2$$

These are eaxctly the same as what we derived in the lecture note. Although the noise here is exponential (which is different from the Gaussian noise assumed in the lecture note), the signals are the same.

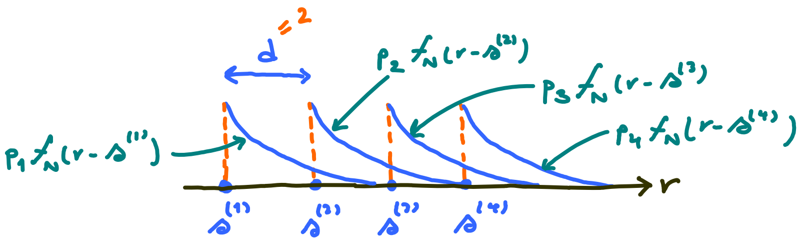
(b) Find the average energy per bit.

Each symbol communicates $\log_2 M = \log_2 4 = 2$ bits.

Therefore, energy per bit $E_b = \frac{E_s}{2} = \frac{5}{8} d^2$

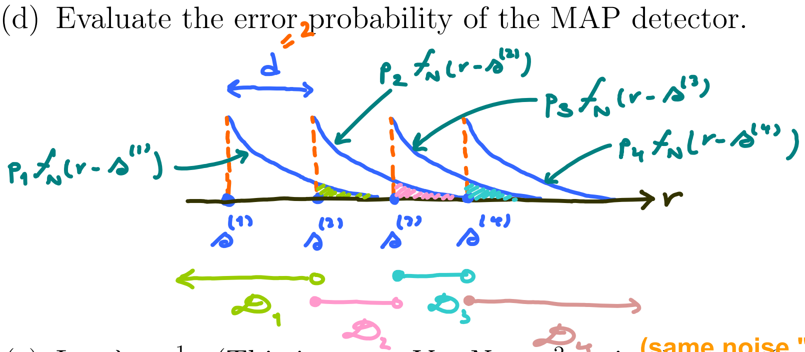
(c) Find the MAP detector $\hat{s}_{\mathrm{MAP}}(r)$.

To find the MAP detector, we compare $p_j f_N(r - s^{(j)})$:

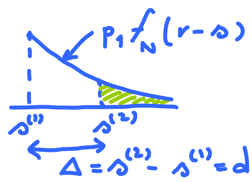


$$\hat{s}_{\mathrm{MAP}}(r) = \begin{cases} s^{(1)}, & r < s^{(2)} \\ s^{(2)}, & s^{(2)} \le r < s^{(3)} \\ s^{(3)}, & s^{(3)} \le r < s^{(4)} \\ s^{(4)}, & s^{(4)} \le r \end{cases} = \begin{cases} -\frac{3d}{2}, & r < -\frac{d}{2} \\ -\frac{d}{2}, & -\frac{d}{2} \le r < \frac{d}{2} \\ \frac{d}{2}, & \frac{d}{2} \le r < \frac{3d}{2} \\ \frac{3d}{2}, & \frac{3d}{2} \le r \end{cases}$$

(d) Evaluate the error probability of the MAP detector.



$P(\mathcal{E}) = 3 \times$



$= 3 \times \frac{1}{4} \times e^{-\lambda d}$

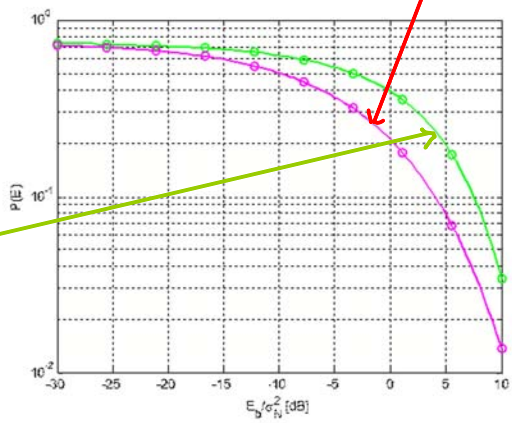
$= \frac{3}{4} e^{-\lambda d}$.

(e) Let $\lambda = \frac{1}{\sigma}$. (This is to set $\mathrm{Var}\, N = \sigma^2$ as in the case for Gaussian noise.) Plot $\frac{E_b}{\sigma^2}$ vs. probability of error $P(\mathcal{E})$. Consider $\frac{E_b}{\sigma^2}$ from -30 to 10 dB.

(same noise "energy")

From part (b), $E_b = \frac{5}{8} d^2$. So, $d = \sqrt{\frac{8}{5} E_b}$.

From part (d), $P(\mathcal{E}) = \frac{3}{4} e^{-\lambda d} = \frac{3}{4} \exp\left(-\frac{1}{\sigma}\sqrt{\frac{8}{5} E_b}\right) = \frac{3}{4} \exp\left(-2\sqrt{\frac{2}{5}\frac{E_b}{\sigma^2}}\right)$



The error probability when the noise is Gaussian (which we derived in the lecture note)

Problem 5. Suppose $s_1(t) = \text{sinc}(5t)$ and $s_2(t) = \text{sinc}(7t)$. Note that in this class, we define $\text{sinc}(x) = \frac{\sin x}{x}$. Find

(a) E_{s_1},

(b) E_{s_2}, and

(c) $\langle s_1(t), s_2(t)\rangle$.

Hint: Evaluate the above quantities in the frequency domain.

From the hint, we will evaluate E_{s_1}, E_{s_2}, and $\langle s_1, s_2\rangle$ in the frequency domain:

$$E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_{-\infty}^{\infty} |S_1(f)|^2 df,$$

$$E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_{-\infty}^{\infty} |S_2(f)|^2 df, \text{ and}$$

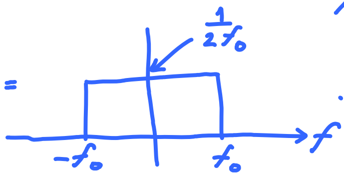
$$\langle s_1, s_2\rangle = \int_{-\infty}^{\infty} s_1(t) s_2^*(t)\, dt = \int_{-\infty}^{\infty} S_1(f) S_2^*(f)\, df.$$

This is not needed here because the waveforms are real-valued.

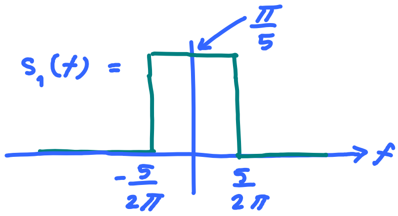
Even though the waveforms are real-valued in time domain, their Fourier transforms may not be real-valued. Therefore, the complex-conjugation is still need here.

Our first step is to find the Fourier transforms $S_1(f)$ and $S_2(f)$ of the waveforms $s_1(t)$ and $s_2(t)$.

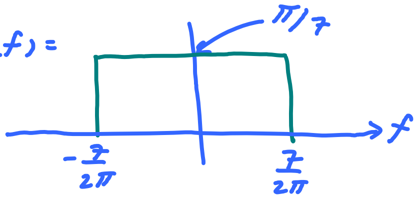
For $g(t) = \text{sinc}(2\pi f_0 t)$, we have seen that $G(f) =$



Therefore, for $s_1(t) = \text{sinc}(5t)$, we have $f_0 = \frac{5}{2\pi}$ and $S_1(f) =$



Similarly, for $s_2(t) = \text{sinc}(7t)$, we have $f_0 = \frac{7}{2\pi}$ and $S_2(f) =$



(a) $E_{s_1} = \int_{-\infty}^{\infty} |S_1(f)|^2 \, df = \left(\frac{\pi}{5}\right)^2 \times \left(2 \times \frac{5}{2\pi}\right) = \frac{\pi}{5}$

(b) $E_{s_2} = \int_{-\infty}^{\infty} |S_2(f)|^2 \, df = \left(\frac{\pi}{7}\right)^2 \times \left(2 \times \frac{7}{2\pi}\right) = \frac{\pi}{7}$

(c) $\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} S_1(f) \, S_2^*(f) \, df = \frac{\pi}{5} \times \frac{\pi}{7} \times 2 \times \frac{5}{2\pi} = \frac{\pi}{7}$

It turns out that $S_2(f)$ is real-valued here. So, it is safe to ignore the complex-conjugation.