ECS 452: Digital Communication Systems	**2015/2**

HW 6 — Due: Apr 27

Lecturer: Asst. Prof. Dr. Prapun Suksompong

Instructions

(a) Solve all non-optional problems. (5 pt)

  (i) Write your first name and the last three digit of your student ID on the upper-right corner of ***every*** submitted page.

  (ii) For each part, write your explanation/derivation and answer **in the space provided**.

(b) ONE part of a question will be graded (5 pt). Of course, you do not know which part will be selected; so you should work on all of them.

(c) Late submission will be rejected.

(d) **Write down all the steps** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider a block code whose generator matrix is

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(a) Suppose the message is $\underline{\mathbf{b}} = [1\,0\,1]$. Find the corresponding codeword $\underline{\mathbf{x}}$.

(b) In the provided table, list all possible data (message) vectors $\underline{\mathbf{b}}$ in the left column (one in each row). Then, find the corresponding codewords $\underline{\mathbf{x}}$ and their weights in the second and third columns, respectively.

$\underline{\mathbf{b}}$	$\underline{\mathbf{x}}$	$w(\underline{\mathbf{x}})$

(c) Find the minimum distance d_{min} for this code.

(d) Suppose we receive $\underline{\mathbf{y}} = [1\,1\,1\,1\,0\,1]$.

(i) Minimum distance decoding:

i. Find the distance $d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ between this received vector $\underline{\mathbf{y}}$ and each of the possible codewords $\underline{\mathbf{x}}$. Put your answers in a new column in the table above.

ii. Use the answer in the previous part to find $\hat{\underline{\mathbf{x}}}$ and $\hat{\underline{\mathbf{b}}}$

(ii) Decoding via the syndrome:

i. Find the parity check matrix $\mathbf{H}$ of this code.

ii. Find the syndrome vector $\underline{\mathbf{s}}$.

iii. Use the answer in the previous parts to find $\hat{\underline{\mathbf{x}}}$ and $\hat{\underline{\mathbf{b}}}$

Problem 2. Consider the following encoding and decoding for a systematic linear block code:

- Encoding
  - The bit positions that are powers of 2 (1, 2, 4, 8, 16, etc.) are check bits.
  - The rest (3, 5, 6, 7, 9, etc.) are filled up with the k data bits.
  - Each check bit forces the parity of some collection of bits, including itself, to be even.
    * To see which check bits the data bit in position i contributes to, rewrite i as a sum of powers of 2. A bit is checked by just those check bits occurring in its expansion.
- Decoding
  - When a codeword arrives, the receiver initializes a counter to zero. It then examines each check bit at position i (i = 1, 2, 4, 8, ...) to see if it has the correct parity.
  - If not, the receiver adds i to the counter. If the counter is zero after all the check bits have been examined (i.e., if they were all correct), the codeword is accepted as valid. If the counter is nonzero, it contains the position of the incorrect bit.

We will consider the case when the codeword's length $n = 7$.

(a) How many bits are check bits?
Hint: How many bit positions are powers of 2?

(b) Find the generator matrix $\mathbf{G}$ for this code.

(c) Find the corresponding parity check matrix **H**.

(d) Explain, from the elements inside the matrix **H**, how this is a Hamming code.

(e) Explain how the decoding instruction above is the “same” as the decoding via the syndrome described in class.

Problem 3. Construct a generator matrix **G** and a corresponding parity check matrix **H** for a (15,11) Hamming code.

Problem 4 (Carlson and Crilly, 2009, P13.2-1)**.** (Optional) In mathematical analysis, a function $d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ is a "true" distance if it satisfies all of the following properties:

(i) positivity: $d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \geq 0$ with equality if and only if $\underline{\mathbf{x}} = \underline{\mathbf{y}}$

(ii) symmetry: $d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) = d(\underline{\mathbf{y}}, \underline{\mathbf{x}})$

(iii) triangle inequality: $d(\underline{\mathbf{x}}, \underline{\mathbf{z}}) \leq d(\underline{\mathbf{x}}, \underline{\mathbf{y}}) + d(\underline{\mathbf{y}}, \underline{\mathbf{z}})$

Is the Hamming distance a "true" distance? (Prove or disprove)

Hint: For the triangle inequality, first consider the number of 1s in $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$, and $\underline{\mathbf{u}} \bigoplus \underline{\mathbf{v}}$ and confirm that $d(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \leq w(\underline{\mathbf{u}}) + w(\underline{\mathbf{v}})$. Then, from this inequality, replace $\underline{\mathbf{u}}$ by $\underline{\mathbf{x}} \bigoplus \underline{\mathbf{y}}$ and $\underline{\mathbf{v}}$ by $\underline{\mathbf{y}} \bigoplus \underline{\mathbf{z}}$.

Problem 5 (Carlson and Crilly, 2009, P13.2-2 and P13.2-3)**.** (Optional) Consider a block code. Suppose $\underline{\mathbf{x}}$ is the transmitted codeword and that $\underline{\mathbf{y}}$ is the vector that results when $\underline{\mathbf{x}}$ is received with $i > 0$ bit errors. Use the triangle inequality for the Hamming distance to show that

(a) if the code has $d_{\min} \geq \ell + 1$ and if $i \leq \ell$, then the errors are detectable.

(b) if the code has $d_{\min} \geq 2t + 1$ and if $i \leq t$, then the errors are correctable by the minimum distance decoder.