ECS 452: Digital Communication Systems **2015/2**

HW 4 — Due: Not Due

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Problem 1 (HW4-2015-2, Free)**.** In each row of the table below, compare the entropy $H(X)$ of the random variable X in the first column with the entropy $H(Y)$ of the random variable Y in the third column by writing ">", "=", or "<" in the second column. Watch out for approximation and round-off error.

Each entropy value can be calculated directly from the given row vectors, Here, we use MATLAB to do this task.

$0.8813 \approx$	$H(X)$ when $\underline{\mathbf{p}} = [0.3, 0.7]$.	$>$	$H(Y)$ when $\underline{\mathbf{q}} = [0.8, 0.2]$.	≈ 0.7219
$1.5710 \approx$	$H(X)$ when $\underline{\mathbf{p}} = [0.3, 0.3, 0.4]$.	$=$	$H(Y)$ when $\underline{\mathbf{q}} = [0.4, 0.3, 0.3]$.	≈ 1.5710
$1.9710 \approx$	$H(X)$ when $p(x) = \begin{cases} 0.3, & x \in \{1, 2\}, \\ 0.2, & x \in \{3, 4\}, \\ 0, & \text{otherwise.} \end{cases}$	$>$	$H(Y)$ when $\underline{\mathbf{q}} = [0.4, 0.3, 0.3]$.	≈ 1.5710

```
function H = entropy2(p)
% ENTROPY2 accepts probability mass function
% as a row vector, calculate the corresponding
% entropy in bits.
p=p(find(abs(sort(p)-1)>1e-8)); % Eliminate 1
p=p(find(abs(p)>1e-8)); % Eliminate 0
if length(p)==0
   H = 0;
else
   H = -sum(p.*log(p))/log(2);
end
```

or see 2.44 in the lecture note.

Problem 2 (HW4-2015-2, Free)**.** Consider random variables X and Y whose joint pmf is given by

The given expression for the joint pmf can be expressed using the joint pmf matrix as

$$p_{X,Y}(x, y) = \begin{cases} c(x + y), & x \in \{1, 3\} \text{ and } y \in \{2, 4\}, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{P} = \begin{array}{c|cc} x \backslash y & 2 & 4 \\ \hline 1 & 3c & 5c \\ 3 & 5c & 7c \end{array}$$

Evaluate the following quantities.

(a) $c = 1/20$

(b) $H(X, Y) = H\Big(\big[\underbrace{\tfrac{3}{20} \ \tfrac{1}{4} \ \tfrac{1}{4} \ \tfrac{7}{20}}_{\text{all elements in the } \mathbf{P} \text{ matrix}}\big]\Big) = -\frac{3}{20}\log_2\frac{3}{20} - \frac{2}{4}\log_2\frac{1}{4} - \frac{7}{20}\log_2\frac{7}{20} \approx 1.9406$ bits.

(c) $H(X) = H\left(\left[\frac{2}{5} \ \frac{3}{5}\right]\right) \approx 0.9710$

(d) $H(Y) = H\left(\left[\frac{2}{5} \ \frac{3}{5}\right]\right) \approx 0.9710$

(e) $H(X|Y) = H(X, Y) - H(Y) \approx 0.9697$

(f) $H(Y|X) = H(X, Y) - H(X) \approx 0.9697$

(g) $I(X; Y) = H(X) + H(Y) - H(X, Y) \approx 0.0013$

The sum of all elements in the P matrix should be 1:

$$3c + 5c + 5c + 7c = 1$$
$$\Rightarrow 20c = 1$$
$$\Rightarrow c = \frac{1}{20}.$$

$$\begin{array}{c|cc|c} x \backslash y & 2 & 4 & p(x) \\ \hline 1 & 3c & 5c & \xrightarrow{\Sigma} 8c = \frac{8}{20} = \frac{2}{5} \\ 3 & 5c & 7c & \xrightarrow{\Sigma} 12c = \frac{12}{20} = \frac{3}{5} \\ \hline & \Sigma\downarrow & \Sigma\downarrow & \\ & 8c & 12c & \\ & \shortparallel & \shortparallel & \\ q(y) & 2/5 & 3/5 & \end{array}$$

Problem 3 (HW4-2015-2, Free)**.** Consider a pair of random variables X and Y whose joint pmf is given by

The given expression for the joint pmf can be expressed using the joint pmf matrix as

$$p_{X,Y}(x,y) = \begin{cases} 1/15, & x = 3, y = 1, \\ 2/15, & x = 4, y = 1, \\ 4/15, & x = 3, y = 3, \\ \beta, & x = 4, y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{P} = \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 3 & 1/15 & 4/15 \\ 4 & 2/15 & \beta \end{array}$$

(a) Find the value of the constant β.

The sum of all elements in the P matrix should be 1:

$$\frac{1}{15} + \frac{4}{15} + \frac{2}{15} + \beta = 1$$

$$\Rightarrow \beta = 1 - \frac{7}{15} = \frac{8}{15}.$$

(b) Are X and Y independent?

$$\mathbf{P} = \begin{array}{c|cc} x \backslash y & 1 & 3 \\ \hline 3 & 1/15 & 4/15 \\ 4 & 2/15 & 8/15 \end{array} \quad \begin{array}{l} \xrightarrow{\Sigma} 5/15 = 1/3 \\ \xrightarrow{\Sigma} 10/15 = 2/3 \end{array} \Bigg\} \Rightarrow \underline{p} = [1/3 \;\; 2/3]$$

$p(x)$

$\downarrow\Sigma$: $3/15 = 1/5$, $\quad 12/15 = 4/5$

$q(y)$: $1/5 \quad 4/5 \Rightarrow \underline{q} = [1/5 \;\; 4/5]$

Recall that two random variables X and Y are indepepndent if and only if

$p(x,y) = p(x)q(y)$ for all pair (x,y)

This is equivalent to $\mathbf{P} = \underline{p}^T \underline{q}$.

In this problem, $\mathbf{P} = \underline{p}^T \underline{q}$. Therefore, $X \perp\!\!\!\perp Y$.

(c) Evaluate the following quantities.

(i) $H(X) = H([1/3 \;\; 2/3]) \approx 0.9183$

(ii) $H(Y) = H([1/5 \;\; 4/5]) \approx 0.7219$

(iii) $H(X,Y) = H(X) + H(Y) \approx 1.6402$

(iv) $H(X|Y) = H(X) \approx 0.9183$

(v) $H(Y|X) = H(Y) \approx 0.7219$

(vi) $I(X;Y) = 0$

These are the results of knowing that X and Y are independent.