HW7 Q1: Fourier Transform Review

Friday, December 05, 2014 1:58 PM

From the hint, we will evaluate E_{s_1}, E_{s_2}, and $\langle s_1, s_2 \rangle$ in the frequency domain:

$$E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_{-\infty}^{\infty} |S_1(f)|^2 df,$$

$$E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_{-\infty}^{\infty} |S_2(f)|^2 df, \quad \text{and}$$

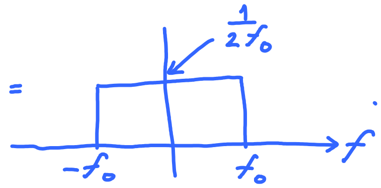
$$\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t) s_2^*(t)\, dt = \int_{-\infty}^{\infty} S_1(f) S_2^*(f)\, df.$$

This is not needed here because the waveforms are real-valued.

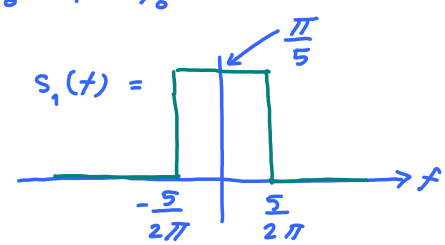
Even though the waveforms are real-valued in time domain, their Fourier transforms may not be real-valued. Therefore, the complex-conjugation is still need here.

Our first step is to find the Fourier transforms $S_1(f)$ and $S_2(f)$ of the waveforms $s_1(t)$ and $s_2(t)$.

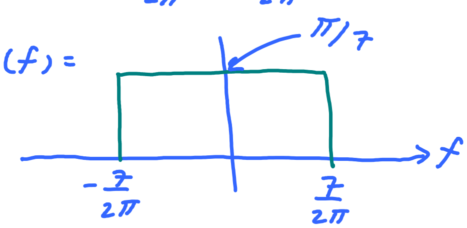
For $g(t) = \text{sinc}(2\pi f_0 t)$, we have seen that $G(f) =$



Therefore, for $s_1(t) = \text{sinc}(5t)$, we have $f_0 = \frac{5}{2\pi}$ and $S_1(f) =$



Similarly, for $s_2(t) = \text{sinc}(7t)$, we have $f_0 = \frac{7}{2\pi}$ and $S_2(f) =$



(a) $E_{s_1} = \int_{-\infty}^{\infty} |S_1(f)|^2 df = \left(\frac{\pi}{5}\right)^2 \times \left(2 \times \frac{5}{2\pi}\right) = \frac{\pi}{5}$

(b) $E_{s_2} = \int_{-\infty}^{\infty} |S_2(f)|^2 df = \left(\frac{\pi}{7}\right)^2 \times \left(2 \times \frac{7}{2\pi}\right) = \frac{\pi}{7}$

(c) $\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} S_1(f) S_2^*(f)\, df = \frac{\pi}{5} \times \frac{\pi}{7} \times 2 \times \frac{5}{2\pi} = \frac{\pi}{7}$

(c) $\langle s_1, s_2 \rangle = \int_{-\infty} S_1(f)\, S_2^*(f)\, df = \frac{\pi}{5} \times \frac{\pi}{7} \times 2 \times \frac{5}{2\pi} = \frac{\pi}{7}$

It turns out that $S_2(f)$ is real-valued here. So, it is safe to ignore the complex-conjugation.

HW7 Q2: 1-D Detector and Uniform Noise

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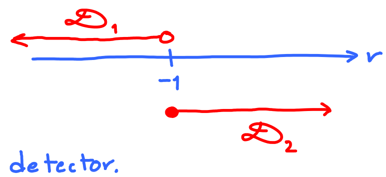
(a) From Q4 of HW6, we found that
$$\hat{s}_{MAP}(r) = \begin{cases} -3, & r < -1 \quad \Rightarrow \mathcal{D}_1 = (-\infty, -1) \\ 3, & r \geq -1 \quad \Rightarrow \mathcal{D}_2 = [-1, \infty) \end{cases}$$
$-3 = s^{(1)}$, $3 = s^{(2)}$

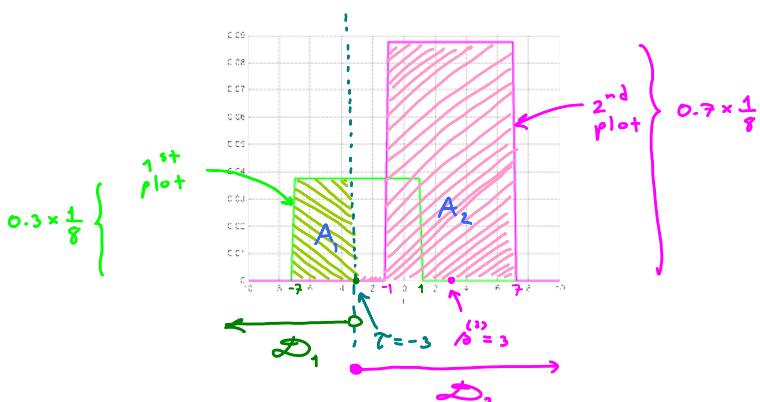


(b) Recall that there are two ways to find $P(\mathcal{E})$ of any detector.

① $$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 1 - \sum_{j=1}^{M} \underbrace{\int_{\mathcal{D}_j} p_j f_N(r - s^{(j)})\, dr}_{\text{The area over } \mathcal{D}_j \text{ under the } j^{th} \text{ plot}}$$

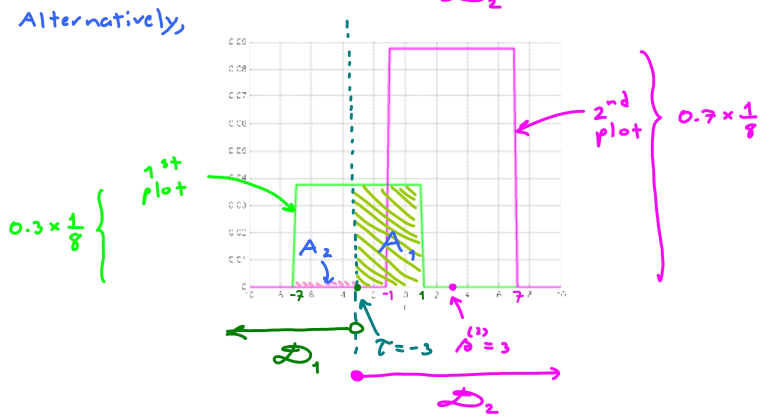
② $$P(\mathcal{E}) = \sum_{j=1}^{M} \overbrace{\int_{\mathcal{D}_j^c} p_j f_N(r - s^{(j)})\, dr}^{\text{The area outside } \mathcal{D}_j \text{ under the } j^{th} \text{ plot}}$$

(b.i) $$\hat{s}(r) = \begin{cases} -3, & r < -3 \\ 3, & r \geq -3 \end{cases}$$



$$P(\mathcal{C}) = A_1 + A_2$$
$$= (-3 - (-7)) \times 0.3 \times \frac{1}{8} + 0.7$$
$$= 0.85$$
$$P(\mathcal{E}) = 1 - 0.85 = 0.15.$$

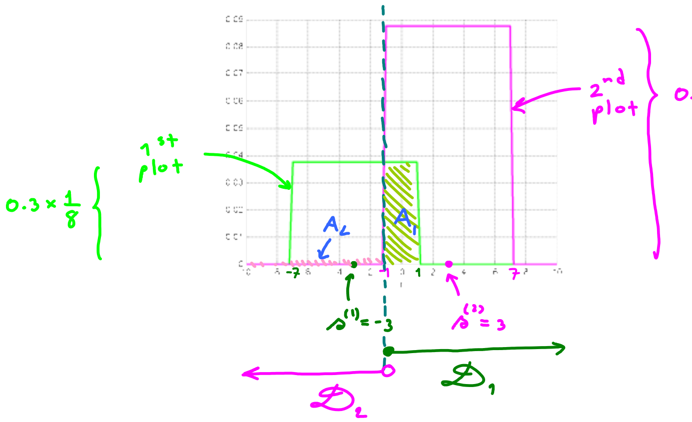
Alternatively,



$$P(\mathcal{E}) = A_1 + A_2$$
$$= (1 - (-3)) \times 0.3 \times \frac{1}{8} + 0$$
$$= 0.15$$

(b.ii) $$\hat{s}(r) = \begin{cases} -3, & r \geq -1 \\ 3, & r < -1 \end{cases}$$

Note that this detector swaps the decision regions of the MAP detector (b.ii) $\hat{s}(r) = \begin{cases} \\ 3, & r < -1 \end{cases}$ decision regions of the MAP detector



$P(\mathcal{C}) = A_1 + A_2$

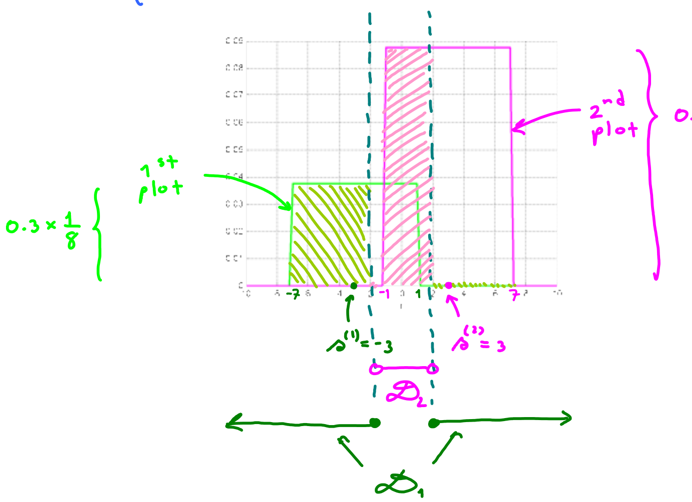
$= (1-(-1)) \times 0.3 \times \frac{1}{8} + 0$

$= \frac{0.3}{4} = 0.075$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 0.9250$

So, it is not surprising that it swaps $P(\mathcal{C})$ and $P(\mathcal{E})$ of the MAP detector.

(b.iii) $\hat{s}(r) = \begin{cases} -3, & |r| \geq 2 \\ 3, & |r| < 2 \end{cases}$



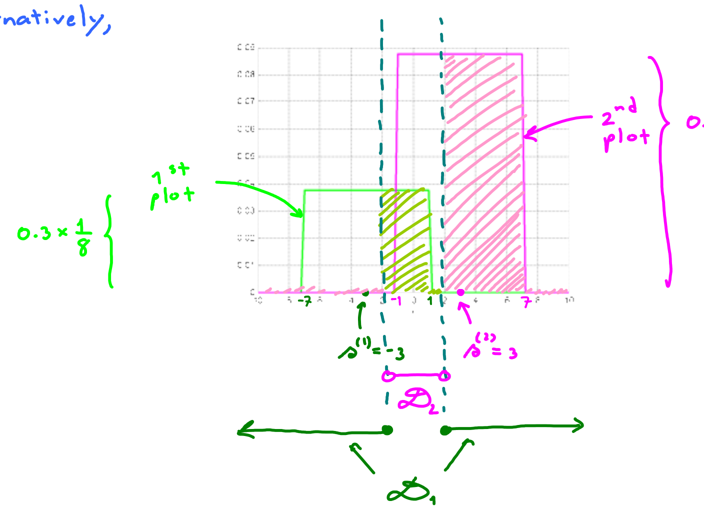
$P(\mathcal{C}) = (-2-(-7)) \times 0.3 \times \frac{1}{8} + (2-(-1)) \times 0.7 \times \frac{1}{8}$

$= \frac{1}{8}(5 \times 0.3 + 3 \times 0.7) = \frac{1}{8}(1.5 + 2.1)$

$= \frac{3.6}{8} = 0.45$

$P(\mathcal{E}) = 1 - P(\mathcal{C}) = 0.55$

Alternatively,



$P(\mathcal{E}) = 0.3 \times \frac{1}{8} \times (1-(-2)) + 0.7 \times \frac{1}{8} \times (7-2)$

$= \frac{1}{8}(0.3 \times 3 + 0.7 \times 5) = \frac{1}{8}(0.9 + 3.5)$

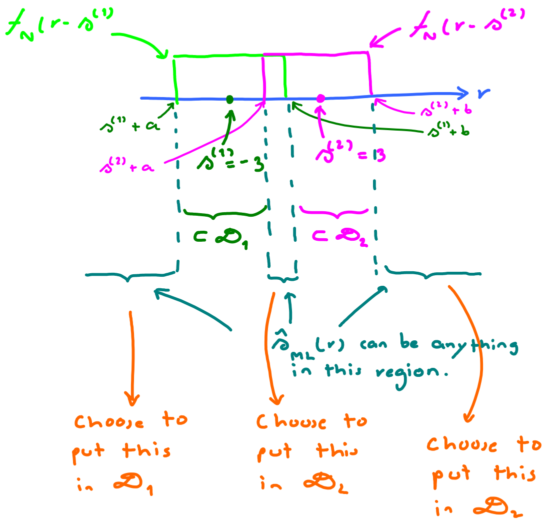
$= \frac{4.4}{8} = 0.55$

(c) Recall that $\hat{s}_{ML}(r) = \arg\max_{s} f_N(r - s)$.

So, the process of finding the ML detector is similar to the one that find the MAP detector except that the shifted noise pdf is not scaled by the prior probabilities.
When there are only two possible s, we compare $f_N(r - s^{(1)})$ and $f_N(r - s^{(2)})$.

$f_N(r - s^{(1)})$ ... $f_N(r - s^{(2)})$ ... $s^{(1)}$ ... $s^{(1)} \ldots r \ldots s^{(2)}$ ... $-3 \ldots r \ldots -1$

When there are only two possible s, we compare $f_N(r-s^{(1)})$ and $f_N(r-s^{(2)})$.



$$\hat{s}_{ML}(r) = \begin{cases} s^{(1)}, & s^{(1)}+a < r < s^{(2)}+a, \\ s^{(2)}, & s^{(1)}+b < r < s^{(2)}+b, \\ \text{anything}, & \text{otherwise} \end{cases} = \begin{cases} -3, & -7 < r < -1, \\ 3, & 1 < r < 7, \\ \text{anything}, & \text{otherwise.} \end{cases}$$

Note that $\hat{s}_{MAP}(r)$ that we found in Q4 of HW6 fits the above conditions.

So, we choose to have $\hat{s}_{ML}(r) = \begin{cases} -3, & r < -1, \\ 3, & r \geq -1. \end{cases}$

Because this is the same as the MAP detector, the error probability must also be the same : $P(\mathcal{E}) = 0.075$

Of course, other choices of ML detector are also possible. The error probability can be found using the techniques similar to part (b).

HW7 Q3: 1-D Multi-Level MAP Detector and Gaussian Noise

Wednesday, July 24, 2013 2:31 PM

Three-point Constellation ($M=3$)

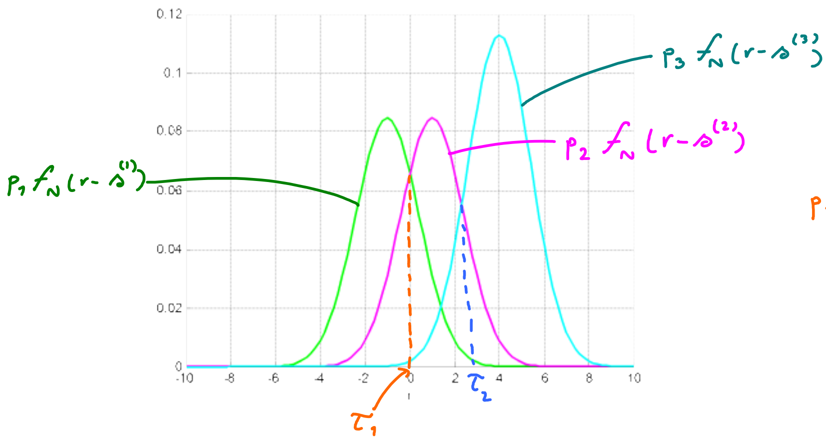
$$S = \{\underset{s^{(1)}}{-1}, \underset{s^{(2)}}{1}, \underset{s^{(3)}}{4}\}$$

$P[S=-1] = 0.3 = P[S=1]$ (p_1, p_2)

$P[S=4] = 0.4$ (p_3)

(a) E_s = Average energy per symbol $= (-1)^2\times 0.3 + 1^2\times 0.3 + 4^2 \times 0.4 = 0.3 + 0.3 + 6.4 = 7$

(b) To find the MAP detector, we compare $p_1 f_N(r-s^{(1)})$, $p_2 f_N(r-s^{(2)})$, and $p_3 f_N(r-s^{(3)})$



To find τ_1, we find r such that

$$p_1 f_N(r-s^{(1)}) = p_2 f_N(r-s^{(2)}).$$

$$p_1 \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{1}{2}\left(\frac{r-s^{(1)}}{\sigma}\right)^2} = p_2 \frac{1}{\sqrt{2\pi}\cancel{\sigma}} e^{-\frac{1}{2}\left(\frac{r-s^{(2)}}{\sigma}\right)^2}$$

$$r = \boxed{\frac{\sigma^2}{s^{(2)}-s^{(1)}} \ln \frac{p_1}{p_2} + \frac{s^{(1)}+s^{(2)}}{2}}$$

This is the same formula that we derived in lecture to find τ for the MAP detector when $M=2$.

Here, $p_1 = p_2$. So, $\tau_1 = \frac{s^{(1)}+s^{(2)}}{2} = \frac{-1+1}{2} = 0$

Similarly, we can find τ_2 by $\frac{\sigma^2}{s^{(3)}-s^{(2)}} \ln \frac{p_3}{p_2} + \frac{s^{(2)}+s^{(3)}}{2} = 2.3082$

The MAP detector is given by

$$\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & r \le \tau_1 \\ s^{(2)}, & \tau_1 < r \le \tau_2 \\ s^{(3)}, & r > \tau_2 \end{cases} = \begin{cases} -1, & r \le 0, \\ 1, & 0 < r \le 2.3082, \\ 4, & r > 2.3082. \end{cases}$$

(c) $\mathcal{D}_1 = \{r : \hat{s}_{MAP}(r) = s^{(1)}\} = (-\infty, 0]$

$\mathcal{D}_2 = \{r : \hat{s}_{MAP}(r) = s^{(2)}\} = (0, 2.3082]$

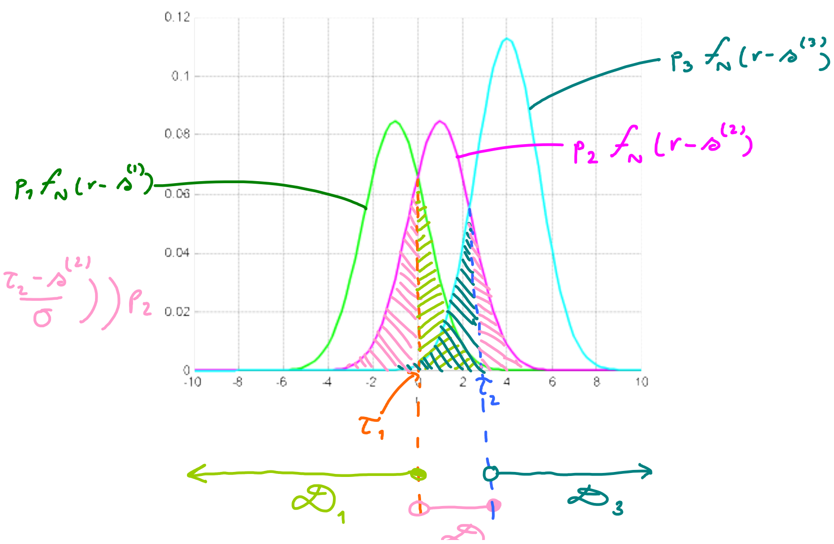
$\mathcal{D}_3 = \{r : \hat{s}_{MAP}(r) = s^{(3)}\} = (2.3082, \infty)$

(d) $$P(\mathcal{E}) = \sum_{j=1}^{M} \int_{\mathcal{D}_j} p_j f_N(r-s^{(j)})\, dr$$

$$= p_1 Q\left(\frac{\tau_1 - s^{(1)}}{\sigma}\right) + \left(Q\left(\frac{s^{(2)}-\tau_1}{\sigma}\right) + Q\left(\frac{\tau_2 - s^{(2)}}{\sigma}\right)\right) p_2$$

$$+ p_3 Q\left(\frac{s^{(3)}-\tau_2}{\sigma}\right)$$

$= 0.2434$

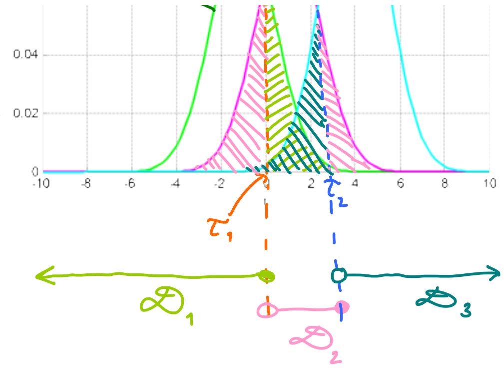


$\mathcal{D}_j$

$$= p_1 Q\left(\frac{\tau_1 - s^{(1)}}{\sigma}\right) + \left(Q\left(\frac{s^{(2)} - \tau_1}{\sigma}\right) + Q\left(\frac{\tau_2 - s^{(2)}}{\sigma}\right)\right) p_2$$

$$+ \; p_3 Q\left(\frac{s^{(3)} - \tau_2}{\sigma}\right)$$

$$= 0.2434$$



HW7 Q4: 1-D Standard Multi-Level MAP Detector and Expo Noise

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(a) Here, there are four ($M=4$) possible values for s : $-\frac{3d}{2}, -\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}$

They are equally likely. Therefore $p_j = \frac{1}{4}$.

$$E_s = \sum_{j=1}^{M} p_j |s^{(j)}|^2 = \frac{1}{4}\left(\left(-\frac{3}{2}d\right)^2 + \left(-\frac{d}{2}\right)^2 + \left(\frac{d}{2}\right)^2 + \left(\frac{3}{2}d\right)^2\right)$$

$$= \frac{1}{16}d^2(9+1+1+9) = \frac{20}{16}d^2 = \frac{5}{4}d^2$$

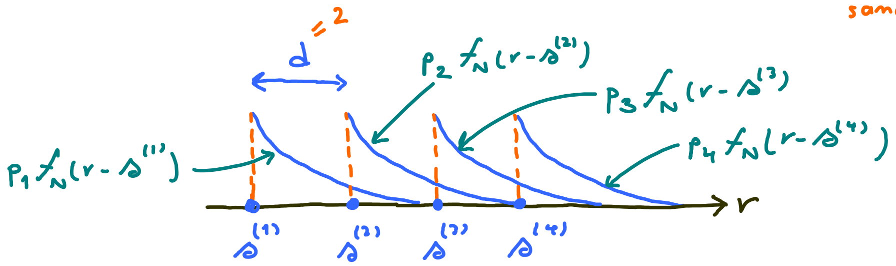
These are exactly the same as what we derived in class.

(b) Each symbol communicates $\log_2 M = \log_2 4 = 2$ bits.

Therefore, energy per bit $E_b = \frac{E_s}{2} = \frac{5}{8}d^2$

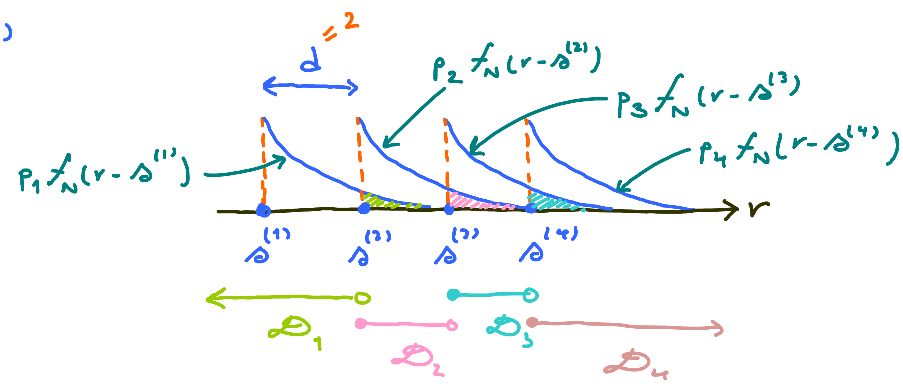
The expo. noise here is different from the Gaussian noise discussed in class but the signals are the same.

(c) To find the MAP detector, we compare $p_j f_N(r - s^{(j)})$:

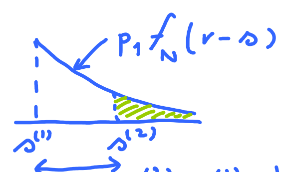


$$\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & r < s^{(2)} \\ s^{(2)}, & s^{(2)} \leq r < s^{(3)} \\ s^{(3)}, & s^{(3)} \leq r < s^{(4)} \\ s^{(4)}, & s^{(4)} \leq r \end{cases} = \begin{cases} -\frac{3d}{2}, & r < -\frac{d}{2} \\ -\frac{d}{2}, & -\frac{d}{2} \leq r < \frac{d}{2} \\ \frac{d}{2}, & \frac{d}{2} \leq r < \frac{3d}{2} \\ \frac{3d}{2}, & \frac{3d}{2} \leq r \end{cases}$$

(d)



$P(\mathcal{E}) = 3 \times$



$$= 3 \times \frac{1}{4} \times e^{-\lambda d}$$

$$= \frac{3}{4} e^{-\lambda d}.$$

(e)

For comparable noise "power" with the Gaussian noise $N \sim \mathcal{N}(0, \sigma^2)$

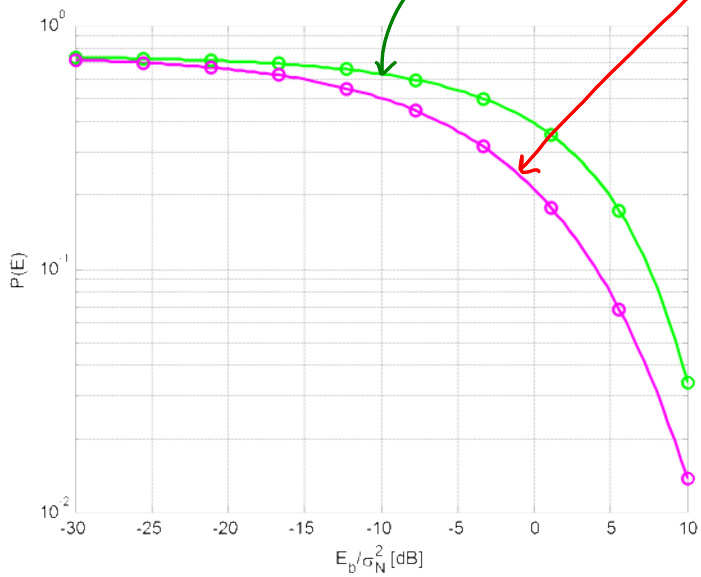
we choose $\text{Var}\, N = \frac{1}{\lambda^2} = \sigma^2 \Rightarrow \frac{1}{\lambda} = \sigma$

mean of the exponential noise

From part (b), $E_b = \frac{5}{8} d^2$. So, $d = \sqrt{\frac{8}{5} E_b}$.

From part (d), $P(\mathcal{E}) = \frac{3}{4} e^{-\lambda d} = \frac{3}{4} \exp\left(-\frac{1}{\sigma}\sqrt{\frac{8}{5} E_b}\right) = \frac{3}{4} \exp\left(-2\sqrt{\frac{2}{5}\frac{E_b}{\sigma^2}}\right)$

This is the plot when the noise is Gaussian (which we derived in class).



HW7 Q5: Hamming Code

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(15, 11) Hamming code

$\left.\begin{array}{l} \hookrightarrow k = 11 \\ \hookrightarrow n = 15 \end{array}\right\} \Rightarrow m = 4.$

First, we construct the parity check matrix H.

$m = 4 \Rightarrow$ Put all $2^4 - 1 = 15$ nonzero 4-bit column vectors as columns of H.

They can be arranged in any order but we must have the identity matrix in the front:

$$\mathbf{H} = \left(\begin{array}{cccc:ccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}\right)$$

I_4 $\quad P$

P^T $\quad I_{11}$

From the recipe discussed in class, when $H = [I \,\vdots\, P]$,

we know that the generator matrix is

$G = [P^T \,\vdots\, I]$

$$\mathbf{G} = \left(\begin{array}{cccc:ccccccccccc}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$