

HW6 Q1 Energy of Modulated Signal

Thursday, December 04, 2014 11:11 PM

$$p(t) = g(t) \cos(2\pi f_c t + \phi) = \frac{g(t)}{2} \left(e^{j(2\pi f_c t + \phi)} + e^{-j(2\pi f_c t + \phi)} \right)$$

\uparrow
 $\cos A = \frac{e^{jA} + e^{-jA}}{2}$

$$= \frac{g(t)}{2} e^{j\phi} e^{j2\pi f_c t} + \frac{g(t)}{2} e^{-j\phi} e^{-j2\pi f_c t}$$

$$\downarrow \mathcal{F}$$

$$P(f) = \frac{e^{j\phi}}{2} G(f-f_c) + \frac{e^{-j\phi}}{2} G(f+f_c)$$

$$E_p = \langle p(t), p(t) \rangle = \langle P(f), P(f) \rangle = \int_{-\infty}^{\infty} P(f) P^*(f) df$$

$$= \int_{-\infty}^{\infty} \left(\underbrace{\frac{e^{j\phi}}{2} G(f-f_c)}_A + \underbrace{\frac{e^{-j\phi}}{2} G(f+f_c)}_B \right) \left(\underbrace{\frac{e^{-j\phi}}{2} G^*(f-f_c)}_C + \underbrace{\frac{e^{j\phi}}{2} G^*(f+f_c)}_D \right) df$$

$$= \int_{-\infty}^{\infty} (A+B)(C+D) df = \int_{-\infty}^{\infty} AC df + \int_{-\infty}^{\infty} AD df + \int_{-\infty}^{\infty} BC df + \int_{-\infty}^{\infty} BD df$$

Now, note that

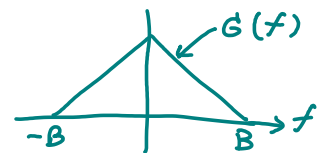
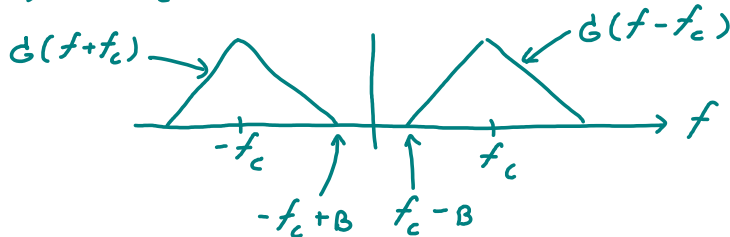
$$AC = \frac{1}{4} |G(f-f_c)|^2$$

$$BD = \frac{1}{4} |G(f+f_c)|^2$$

suppose we assume that $G(f)$ is band-limited to $\pm B$, i.e.,

$$G(f) = 0 \text{ when } |f| > B.$$

Then, if $f_c > B$,



the non-zero parts of $G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Therefore, $\left. \begin{matrix} BC \equiv 0 \\ AD \equiv 0 \end{matrix} \right\}$ across all f .

$$\text{so, } E_p = \int_{-\infty}^{\infty} \frac{1}{4} |G(f-f_c)|^2 df + 0 + 0 + \int_{-\infty}^{\infty} \frac{1}{4} |G(f+f_c)|^2 df$$

$$\begin{aligned}
 \text{so, } E_p &= \int_{-\infty}^{\infty} \frac{1}{4} |G(f-f_c)|^2 df + 0 + 0 + \int_{-\infty}^{\infty} \frac{1}{4} |G(f+f_c)|^2 df \\
 &= \int_{-\infty}^{\infty} \frac{1}{4} |G(u)|^2 du + \int_{-\infty}^{\infty} \frac{1}{4} |G(u)|^2 du \quad \begin{array}{l} \text{(change of} \\ \text{variables)} \end{array} \\
 &= \frac{1}{4} E_g + \frac{1}{4} E_g = \frac{E_g}{2}.
 \end{aligned}$$

This formula is valid when $G(f) = 0$ for $|f| \geq f_c$.

HW6 Q2: Average Symbol Energy

Thursday, December 4, 2014 11:30 PM

Recall that $E_{\vec{s}} = \sum_{j=1}^M p_j E_j$ where $E_j = \text{energy of } \vec{s}_j(t) = \langle \vec{s}_j(t), \vec{s}_j(t) \rangle$
 $= \text{energy of } \vec{s}^{(j)} = \langle \vec{s}^{(j)}, \vec{s}^{(j)} \rangle$

There are three possible \vec{s}
 $\hookrightarrow \vec{s}^{(1)}, \vec{s}^{(2)}, \vec{s}^{(3)}$

Here, $M = 3$ and $K = 1$.

\vec{s} is one-dimensional
(a scalar)

$$= \sum_{i=1}^K |\underline{s}_i^{(j)}|^2$$

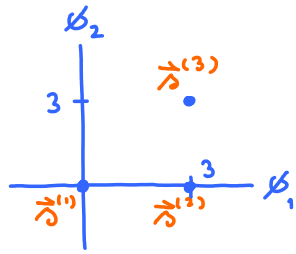
↑
the i^{th} element
in vector $\vec{s}^{(j)}$

$$\text{so, } E_j = |\vec{s}^{(j)}|^2 \quad \text{and} \quad E_s = \sum_{j=1}^3 p_j |\vec{s}^{(j)}|^2$$
$$= 0.41 \times (-1)^2 + 0.08 \times (1)^2 + 0.51 \times 4^2 = 8.65$$

HW6 Q3: Minimum Energy for Constellation

Thursday, September 05, 2013 3:31 PM

(a)



$$\vec{d}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{d}^{(2)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\vec{d}^{(3)} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$E_s = \frac{1}{3} (0^2 + 3^2 + 3^2 + 3^2) = 9$$

* dimensions. Here, $K = 2$.

$$(b) E_s = \sum_{j=1}^M p_j E_j = \sum_{j=1}^M p_j \sum_{i=1}^K |\vec{d}_i^{(j)}|^2 = \sum_{i=1}^K \sum_{j=1}^M p_j |\vec{d}_i^{(j)}|^2$$

energy of $\vec{d}^{(j)}$
probability of $\vec{d}^{(j)}$
the i th element in the vector $\vec{d}^{(j)}$

$$= \frac{1}{3} \left((0-a_1)^2 + (2-a_1)^2 + (2-a_1)^2 \right) + \frac{1}{3} \left((0-a_2)^2 + (0-a_2)^2 + (2-a_2)^2 \right)$$

In general, we have to minimize terms of the form $\sum_{j=1}^M p_j (x_j - a)^2$.

Method 1:

$$\begin{aligned} \sum_{j=1}^M p_j (x_j - a)^2 &= \mathbb{E}[(X - a)^2] = \mathbb{E}[(X - \underbrace{\mathbb{E}X}_0 + \mathbb{E}X - a)^2] \\ &= \text{Var } X + 2 \mathbb{E}[(X - \mathbb{E}X)] (\mathbb{E}X - a) + (\mathbb{E}X - a)^2 \\ &= \text{Var } X + \underbrace{(\mathbb{E}X - a)^2} \end{aligned}$$

This is the only term that depends on a .
Minimum value of 0 is achieved when $a = \mathbb{E}X$.

Method 2:

$$\sum_{j=1}^M p_j (x_j - a)^2 = \sum_{j=1}^M p_j (x_j^2 - 2ax_j + a^2) = a^2 - 2a \mathbb{E}X + \mathbb{E}[x^2]$$

As a function of " a ", this is a parabola with minimum at

$$a = \mathbb{E}X.$$

$$a = \mathbb{E}X.$$

set $\frac{d}{da}(\) = 0$. Then solve for a .

Method 3:

$$\frac{d}{da} \sum_{j=1}^M p_j (x_j - a)^2 = \sum_{j=1}^M p_j 2(x_j - a) = 2(\mathbb{E}X - a)$$

The derivative above is 0 when $a = \mathbb{E}X$.

So, the minimum value occurs when $a = \mathbb{E}X = \sum_i p_i x_i$

Minimum E_s occurs when

$$a_1 = \frac{1}{3}(0+3+3) = 2$$

The first elements
in the vectors.

$$a_2 = \frac{1}{3}(0+0+3) = 1.$$

The second elements
in the vectors.

HW6 Q4: 1-D MAP Detector and Uniform Noise

Monday, July 15, 2013 1:33 PM

Recall that the uniform pdf on $[a, b]$ is given by

$$f_N(n) = \begin{cases} \frac{1}{b-a}, & a < n < b, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $a=4$ and $b=-4$. So, $f_N(n) = \begin{cases} 1/8, & -4 < n < 4, \\ 0, & \text{otherwise.} \end{cases}$

(a) Again, the MAP detector is given by

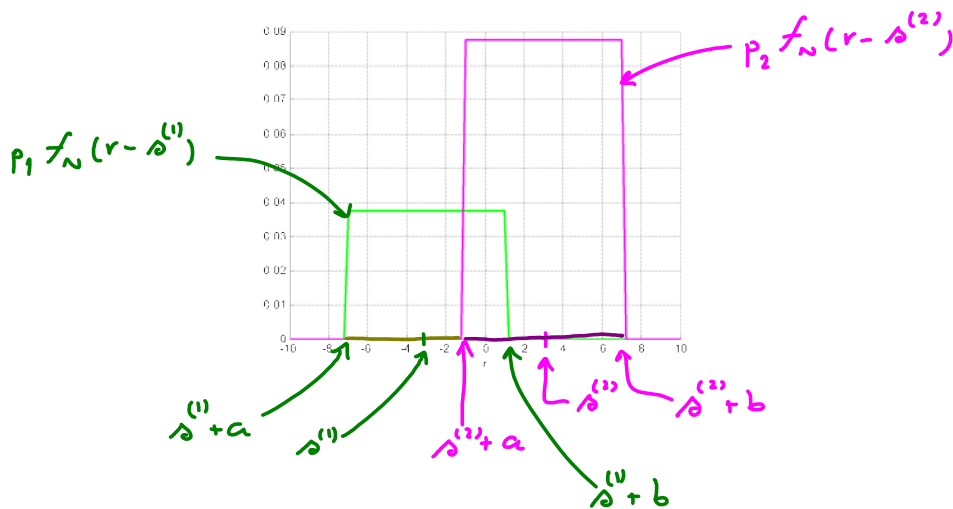
$$\hat{s}_{MAP}(r) = \arg \max_s p_s f_N(r-s).$$

This is true regardless of the pdf of the noise

Here, there are two possible values for s .

So we compare $p_1 f_N(r-s^{(1)})$ and $p_2 f_N(r-s^{(2)})$

\uparrow \uparrow \uparrow \uparrow
 0.3 -3 0.7 3



Observe that

① When $s^{(1)}+a < r < s^{(2)}+a$, $p_1 f_N(r-s^{(1)}) > p_2 f_N(r-s^{(2)})$.

Therefore, $\hat{s}_{MAP}(r) = s^{(1)}$ in this region.

② When $s^{(1)}+a < r < s^{(2)}+b$, $p_1 f_N(r-s^{(1)}) < p_2 f_N(r-s^{(2)})$.

Therefore, $\hat{\Delta}_{MAP}(r) = \Delta^{(2)}$ in this region.

③ When $r < \Delta^{(1)} + a$ or $r > \Delta^{(2)} + b$, the pdf in both cases are 0. So, these are the impossible regions. The received signal R won't fall in these regions. Therefore, it does not matter how the detector behaves in this region.

$$\text{Conclusion: } \hat{\Delta}_{MAP}(r) = \begin{cases} \Delta^{(1)}, & \Delta^{(1)} + a < r < \Delta^{(2)} + a, \\ \Delta^{(2)}, & \Delta^{(2)} + a < r < \Delta^{(2)} + b, \\ \text{anything,} & \text{otherwise} \end{cases}$$

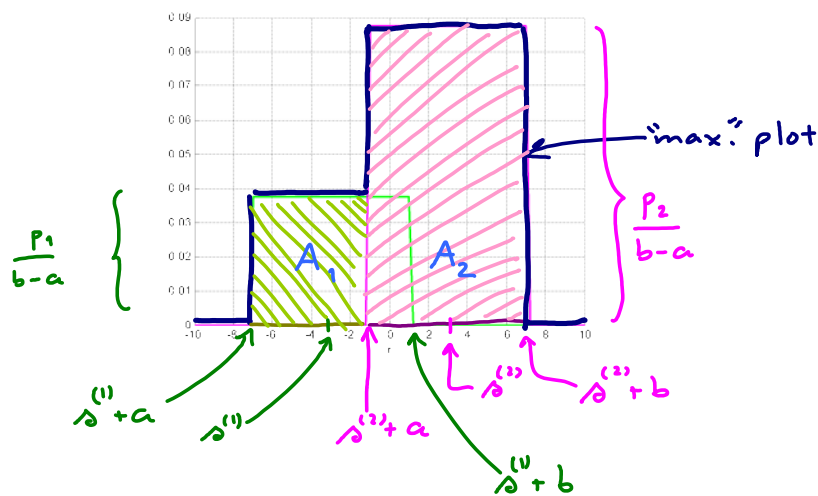
Note that at $r = \Delta^{(1)} + a, \Delta^{(2)} + a, \Delta^{(2)} + b$ (the boundaries), the comparison of $p_1 f_N(r - \Delta^{(1)})$ and $p_2 f_N(r - \Delta^{(2)})$ depends on how the pdf of noise is defined at its boundaries. However, the probability that the received signal R will be exactly any one of these three points is zero. Therefore, we can define $\hat{\Delta}_{MAP}(r)$ to be anything here.

To further simplify the expression, we choose the "anything" parts above in a way that they can be combined into adjacent intervals. This gives

$$\hat{\Delta}_{MAP}(r) = \begin{cases} \Delta^{(1)}, & r < \Delta^{(2)} + a \\ \Delta^{(2)}, & r \geq \Delta^{(2)} + a \end{cases} = \begin{cases} -3, & r < -1 \\ 3, & r \geq -1 \end{cases}$$

$\tau^* = \Delta^{(2)} + a$
 $= 3 + (-4)$
 $= -1$

(b) Recall that for MAP detector, $P(C) = \text{area under the "max." plot.}$



$$\text{Area } A_2 = \frac{P_1}{b-a} \times ((\Delta^{(2)} + a) - (\Delta^{(1)} + a)) = \frac{P_1}{b-a} (\Delta^{(2)} - \Delta^{(1)})$$

NOTO

$$\text{Area } A_1 = \frac{p_1}{b-a} \times ((s^{(2)}+a) - (s^{(1)}+a)) = \frac{p_1}{b-a} (s^{(2)} - s^{(1)})$$

$$\text{Area } A_2 = \frac{p_2}{b-a} \times ((s^{(2)}+b) - (s^{(1)}+a)) = p_2$$

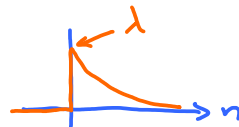
$$\text{So, } P(C) = \frac{p_1}{b-a} (s^{(2)} - s^{(1)}) + p_2.$$

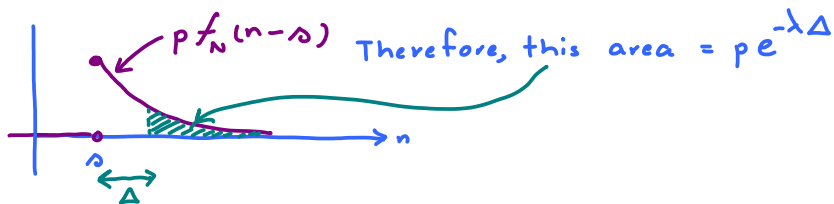
$$\begin{aligned} \text{Therefore, } P(E) &= 1 - P(C) = p_1 - \frac{p_1}{b-a} (s^{(2)} - s^{(1)}) = p_1 \left(1 - \frac{s^{(2)} - s^{(1)}}{b-a} \right) \\ &= 0.3 \left(1 + \frac{3 - (-3)}{4 - (-4)} \right) = 0.3 \left(1 - \frac{3}{4} \right) = 0.3 \times \frac{1}{4} = 0.075 \end{aligned}$$

HW6 Q5: 1-D MAP Detector and Exponential Noise

Monday, July 15, 2013 10:58 AM

Some facts about exponential noise:

- ① $f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$ 
- ② $E[N] = \frac{1}{\lambda}$ and $\text{Var} N = \frac{1}{\lambda^2}$
- ③ MATLAB use $E[N]$ as the parameter instead of λ
- ④ $P[N > n] = \int_n^{\infty} f_N(n) dn = \begin{cases} e^{-\lambda n}, & n \geq 0 \\ 1, & n < 0 \end{cases}$

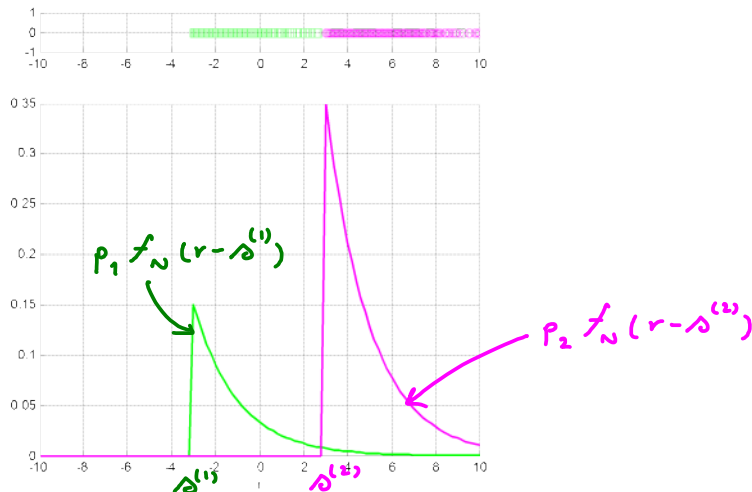


In this question, $\lambda = \frac{1}{2}$ and $E[N] = \frac{1}{\lambda} = 2$.

(a) MAP Detector: Recall that $\hat{\Delta}_{MAP}(r) = \arg \max_{\Delta} P_{\Delta} f_N(r - \Delta)$
 This is true regardless of the pdf of the noise

Here, there are two possible values for Δ .

So we compare $P_1 f_N(r - \Delta^{(1)})$ and $P_2 f_N(r - \Delta^{(2)})$
 $\uparrow \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow$
 0.3 \quad -3 \quad 0.7 \quad 3



From the graph, it is clear that

① $R < \delta^{(1)}$ is impossible. So, the detector can do anything in this region without affecting its performance.

② When $\delta^{(1)} < r < \delta^{(2)}$, $p_1 f_N(r - \delta^{(1)}) > p_2 f_N(r - \delta^{(2)})$.
So, in this region, $\hat{\delta}_{MAP}(r) = \delta^{(1)}$.

③ When $r > \delta^{(2)}$, $p_1 f_N(r - \delta^{(1)}) < p_2 f_N(r - \delta^{(2)})$.
So, in this region, $\hat{\delta}_{MAP}(r) = \delta^{(2)}$.

Conclusion:

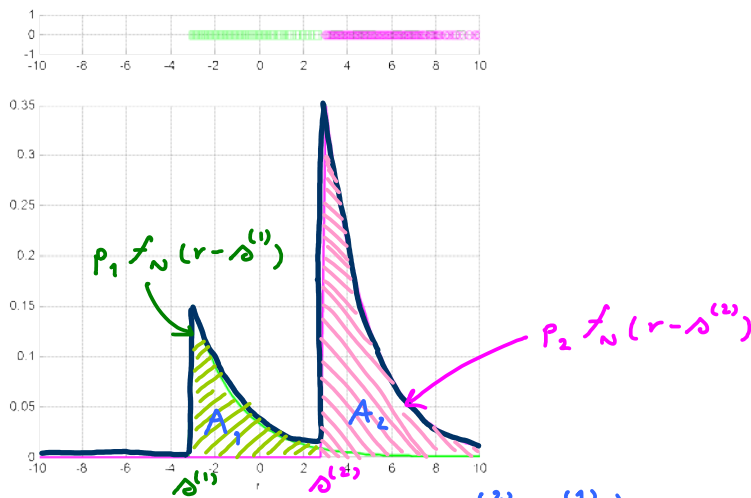
$$\hat{\delta}_{MAP}(r) = \begin{cases} \delta^{(1)}, & \delta^{(1)} < r < \delta^{(2)}, \\ \delta^{(2)}, & r > \delta^{(2)}, \\ \text{anything, otherwise} & \end{cases} = \begin{cases} \delta^{(1)}, & r < \delta^{(2)} \\ \delta^{(2)}, & r \geq \delta^{(2)} \end{cases}$$

Simplification

$$= \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

$$\begin{aligned} r < \delta^{(2)} \\ r \geq \delta^{(2)} \\ \uparrow \\ r^* = \delta^{(2)} \\ = 3 \end{aligned}$$

(b) Recall that for MAP detector, $P(C) = \text{area under the "max." plot.}$



$$\text{Area } A_1 = p_1 - p_1 e^{-\lambda(\delta^{(2)} - \delta^{(1)})}$$

$$\text{Area } A_2 = p_2$$

$$\text{So, } P(C) = 1 - p_1 e^{-\lambda(\delta^{(2)} - \delta^{(1)})} \text{ and}$$

$$\begin{aligned} P(E) = 1 - P(C) &= p_1 e^{-\lambda(\delta^{(2)} - \delta^{(1)})} = 0.3 e^{-\frac{1}{2}(3 - (-3))} \\ &= 0.3 e^{-3} \approx 0.0149 \end{aligned}$$