HW6 Q1 Energy of Modulated Signal

Thursday, December 04, 2014 11:11 PM

$$p(t) = g(t)\cos(2\pi f_c t + \phi) = \frac{g(t)}{2}\left(e^{j(2\pi f_c t + \phi)} + e^{-j(2\pi f_c t + \phi)}\right)$$

$$\cos A = \frac{e^{jA} + e^{-jA}}{2}$$

$$= \frac{g(t)}{2} e^{j\phi} e^{j2\pi f_c t} + \frac{g(t)}{2} e^{-j\phi} e^{-j2\pi f_c t}$$

$$\Big\downarrow \mathcal{F}$$

$$P(f) = \frac{e^{j\phi}}{2} G(f-f_c) + \frac{e^{-j\phi}}{2} G(f+f_c)$$

$$E_p = \langle p(t), p(t)\rangle = \langle P(f), P(f)\rangle = \int_{-\infty}^{\infty} P(f)\, P^*(f)\, df$$

$$= \int_{-\infty}^{\infty} \Big(\underbrace{\frac{e^{j\phi}}{2} G(f-f_c)}_{A} + \underbrace{\frac{e^{-j\phi}}{2} G(f+f_c)}_{B}\Big)\Big(\underbrace{\frac{e^{-j\phi}}{2} G^*(f-f_c)}_{C} + \underbrace{\frac{e^{j\phi}}{2} G^*(f+f_c)}_{D}\Big) df$$

$$= \int_{-\infty}^{\infty} (A+B)(C+D)\, df = \int_{-\infty}^{\infty} AC\, df + \int_{-\infty}^{\infty} AD\, df + \int_{-\infty}^{\infty} BC\, df + \int_{-\infty}^{\infty} BD\, df$$

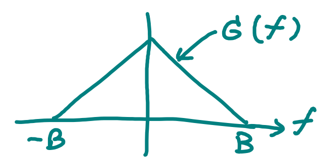
Now, note that

$$AC = \frac{1}{4}|G(f-f_c)|^2$$

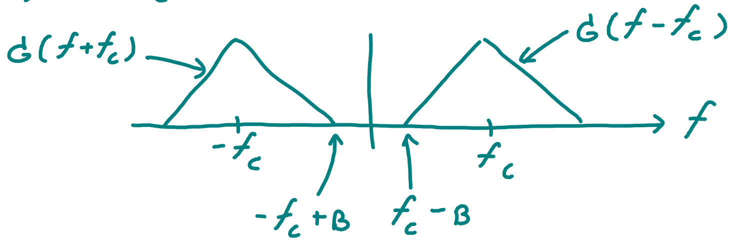
$$BD = \frac{1}{4}|G(f+f_c)|^2$$

suppose we assume that $G(f)$ is band-limited to $\pm B$, i.e.,

$$G(f) = 0 \quad \text{when} \quad |f| > B.$$



Then, if $f_c > B$,



the non-zero parts of $G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Therefore, $BC \equiv 0$, $AD \equiv 0$ } across all f.

$$\text{So,}\quad E_p = \int_{-\infty}^{\infty} \frac{1}{4}|G(f-f_c)|^2 df + 0 + 0 + \int_{-\infty}^{\infty} \frac{1}{4}|G(f+f_c)|^2 df$$

So, $E_p = \int_{-\infty} \frac{1}{4}|G(f-f_c)|^2 df + 0 + 0 + \int_{-\infty} \frac{1}{4}|G(f+f_c)|^2 df$

$\mu = f - f_c$ $\qquad$ $\mu = f + f_c$ $\qquad$ (change of variables)

$= \int_{-\infty}^{\infty} \frac{1}{4}|G(\mu)|^2 d\mu + \int_{-\infty}^{\infty} \frac{1}{4}|G(\mu)|^2 d\mu$

$= \frac{1}{4}E_g + \frac{1}{4}E_g = \frac{E_g}{2}.$

This formula is valid when $G(f) = 0$ for $|f| \geq f_c$.

HW6 Q2: Average Symbol Energy

Thursday, December 4, 2014 11:30 PM

Recall that $E_s = \sum_{j=1}^{M} p_j E_j$ where E_j = energy of $s_j(t) = \langle s_j(t), s_j(t) \rangle$

$= \text{energy of } \vec{s}^{(j)} = \langle \vec{s}^{(j)}, \vec{s}^{(j)} \rangle$

$= \sum_{i=1}^{K} |\underbrace{s_i^{(j)}}|^2$

the i^{th} element in vector $\vec{s}^{(j)}$

There are three possible $\vec{s}$

$\hookrightarrow \vec{s}^{(1)}, \vec{s}^{(2)}, \vec{s}^{(3)}$

Here, $M = 3$ and $K = 1$,

$\vec{s}$ is one-dimensional (a scalar)

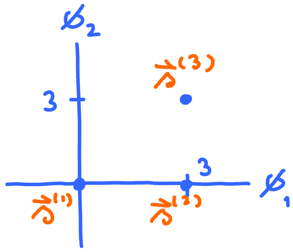
So, $E_j = |s^{(j)}|^2$ and $E_s = \sum_{j=1}^{3} p_j |s^{(j)}|^2$

$= 0.41 \times (-1)^2 + 0.08 \times (1)^2 + 0.51 \times 4^2 = 8.65$

HW6 Q3: Minimum Energy for Constellation

Thursday, September 05, 2013 3:31 PM

(a)



$$\vec{s}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{s}^{(2)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \vec{s}^{(3)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

$$E_s = \frac{1}{3}\left(0^2 + 3^2 + 3^2 + 3^2\right) = 9$$

(b)
$$E_s = \sum_{j=1}^{M} \underbrace{p_j}_{\text{probability of } \vec{s}^{(j)}} \underbrace{E_j}_{\text{energy of } \vec{s}^{(j)}} = \sum_{j=1}^{M} p_j \sum_{i=1}^{K} \left|\underbrace{\vec{s}_i^{(j)}}_{\text{the } i^{th} \text{ element in the vector } \vec{s}^{(j)}}\right|^2 = \sum_{i=1}^{2} \sum_{j=1}^{M} p_j \left|\vec{s}_i^{(j)}\right|^2$$

K: # dimensions. Here, K = 2.

$$= \frac{1}{3}\left((0-a_1)^2 + (2-a_1)^2 + (2-a_1)^2\right) + \frac{1}{3}\left((0-a_2)^2 + (0-a_2)^2 + (2-a_2^2)\right)$$

In general, we have to minimize terms of the form $\sum_{j=1}^{M} p_j (x_j - a)^2$.

Method 1:
$$\sum_{j=1}^{M} p_j (x_j - a)^2 = \mathbb{E}\left[(X-a)^2\right] = \mathbb{E}\left[(X - \underbrace{\mathbb{E}X + \mathbb{E}X}_{0} - a)^2\right]$$

$$= \operatorname{Var} X + 2\,\underbrace{\mathbb{E}\left[(X - \mathbb{E}X)\right]}_{0}(\mathbb{E}X - a) + (\mathbb{E}X - a)^2$$

$$= \operatorname{Var} X + \underbrace{(\mathbb{E}X - a)^2}$$

This is the only term that depends on a. Minimum value of 0 is achieved when $a = \mathbb{E}X$.

Method 2:
$$\sum_{j=1}^{M} p_j (x_j - a)^2 = \sum_{j=1}^{M} p_j (x_j^2 - 2a x_j + a^2) = a^2 - 2a\,\mathbb{E}X + \mathbb{E}[X^2]$$

As a function of "a", this is a parabola $\cup$ with minimum at $a = \mathbb{E}X$.

$a = \mathbb{E}X.$ set $\frac{d}{da}(\quad) = 0$. Then solve for a.

Method 3:

$$\frac{d}{da}\sum_{j=1}^{M} p_j (x_j - a)^2 = \sum_{j=1}^{M} p_j\, 2(x_j - a) = 2(\mathbb{E}X - a)$$

The derivative above is 0 when $a = \mathbb{E}X$.

So, the minimum value occurs when $a = \mathbb{E}X = \sum_i p_i x_i$

Minimum E_s occurs when

$a_1 = \frac{1}{3}(0 + 3 + 3) = 2$ $\qquad$ $a_2 = \frac{1}{3}(0 + 0 + 3) = 1$.

The first elements in the vectors. $\qquad$ The second elements in the vectors.

HW6 Q4: 1-D MAP Detector and Uniform Noise

Monday, July 15, 2013 1:33 PM

Recall that the uniform pdf on $[a,b]$ is given by

$$f_N(n) = \begin{cases} \frac{1}{b-a}, & a < n < b, \\ 0, & \text{otherwise.} \end{cases}$$

Here, $a = 4$ and $b = -4$. So, $f_N(n) = \begin{cases} 1/8, & -4 < n < 4, \\ 0 & \text{otherwise.} \end{cases}$

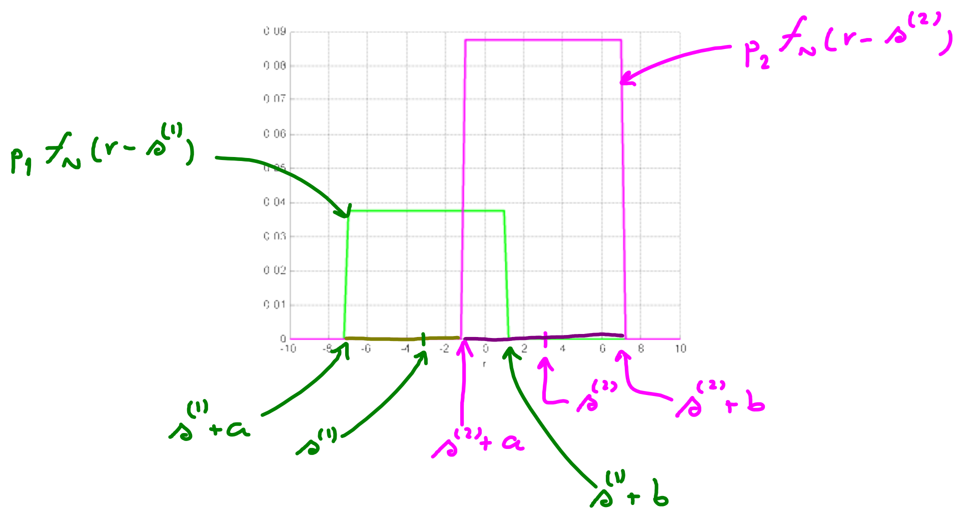
(a) Again, the MAP detector is given by

$$\hat{s}_{MAP}(r) = \arg\max_{s} p_s f_N(r-s).$$

This is true regardless of the pdf of the noise

Here, there are two possible values for s.

So we compare $\underset{0.3}{p_1}\, f_N(r - \underset{-3}{s^{(1)}})$ and $\underset{0.7}{p_2}\, f_N(r - \underset{3}{s^{(2)}})$



Observe that

① when $s^{(1)}+a < r < s^{(2)}+a$, $p_1 f_N(r-s^{(1)}) > p_2 f_N(r-s^{(2)})$.

Therefore, $\hat{s}_{MAP}(r) = s^{(1)}$ in this region.

② when $s^{(2)}+a < r < s^{(2)}+b$, $p_1 f_N(r-s^{(1)}) < p_2 f_N(r-s^{(2)})$.

Therefore, $\hat{s}_{MAP}(r) = s^{(2)}$ in this region.

③ When $r < s^{(1)}+a$ or $r > s^{(2)}+b$, the pdf in both cases are 0. So, these are the impossible regions. The received signal R won't fall in these regions. Therefore, it does not matter how the detector behaves in this region.

$$\text{Conclusion:} \quad \hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & s^{(1)}+a < r < s^{(2)}+a, \\ s^{(2)}, & s^{(2)}+a < r < s^{(2)}+b, \\ \text{anything}, & \text{otherwise} \end{cases}$$

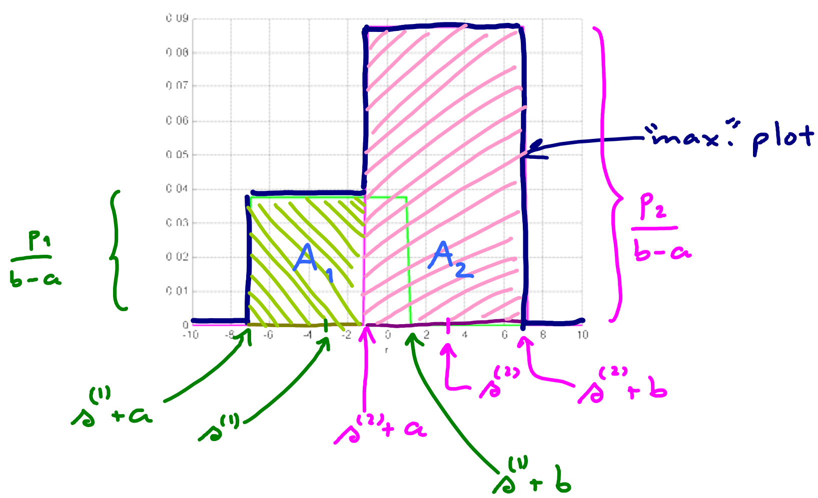
Note that at $r = s^{(1)}+a,\ s^{(2)}+a,\ s^{(2)}+b$ (the boundaries), the comparison of $p_1 f_N(r-s^{(1)})$ and $p_2 f_N(r-s^{(2)})$ depends on how the pdf of noise is defined at its boundaries. However, the probability that the received signal R will be exactly any one of these three points is zero. Therefore, we can define $\hat{s}_{MAP}(r)$ to be anything here.

To further simply the expression, we choose the "anything" parts above in a way that they can be combined into adjacent intervals. This gives

$$\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & r < s^{(2)}+a \\ s^{(2)}, & r \geq \underbrace{s^{(2)}+a}_{\tau^*} \end{cases} = \begin{cases} -3, & r < -1 \\ 3, & r \geq -1 \end{cases}$$

$$\tau^* = s^{(2)}+a = 3+(-4) = -1$$

(b) Recall that for MAP detector, $P(\mathcal{C})$ = area under the "max." plot.



$$\text{Area } A_1 = \frac{p_1}{b-a} \times \left((s^{(2)}+\cancel{a}) - (s^{(1)}+\cancel{a}) \right) = \frac{p_1}{b-a}\left(s^{(2)}-s^{(1)}\right)$$

$s + b$

Area $A_1 = \frac{p_1}{b-a} \times \left((s^{(2)} + \cancel{a}) - (s^{(1)} + \cancel{a}) \right) = \frac{p_1}{b-a} \left(s^{(2)} - s^{(1)} \right)$

Area $A_2 = \frac{p_2}{b-a} \times \left((\cancel{s^{(2)}} + b) - (\cancel{s^{(2)}} + a) \right) = p_2$

So, $P(C) = \frac{p_1}{b-a} \left(s^{(2)} - s^{(1)} \right) + p_2$.

Therefore, $P(\mathcal{E}) = 1 - P(C) = p_1 - \frac{p_1}{b-a} \left(s^{(2)} - s^{(1)} \right) = p_1 \left(1 - \frac{s^{(2)} - s^{(1)}}{b-a} \right)$

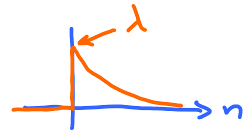
$$= 0.3 \left(1 + \frac{3-(-3)}{4-(-4)} \right) = 0.3 \left(1 - \frac{3}{4} \right) = 0.3 \times \frac{1}{4} = 0.075$$

HW6 Q5: 1-D MAP Detector and Exponential Noise

Monday, July 15, 2013 10:58 AM

Some facts about exponential noise:

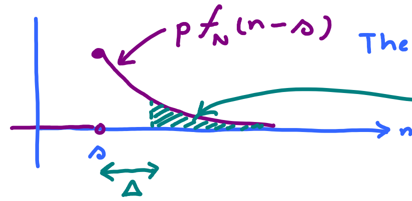
1. $f_N(n) = \begin{cases} \lambda e^{-\lambda n}, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$



2. $\mathbb{E}N = \frac{1}{\lambda}$ and $\mathrm{Var}\, N = \frac{1}{\lambda^2}$

3. MATLAB use $\mathbb{E}N$ as the parameter instead of λ

4. $P[N > n] = \int_n^\infty f_N(n)\, dn = \begin{cases} e^{-\lambda n}, & n \geq 0 \\ 1, & n < 0 \end{cases}$



Therefore, this area $= p e^{-\lambda \Delta}$

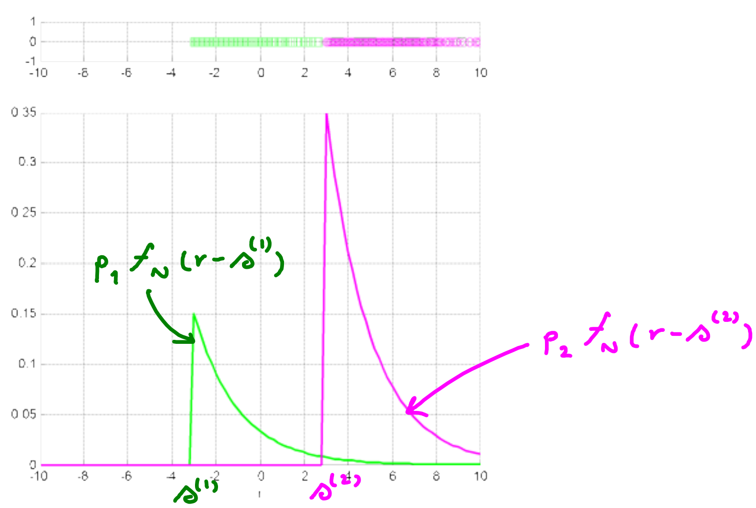
In this question, $\lambda = \frac{1}{2}$ and $\mathbb{E}N = \frac{1}{\lambda} = 2$.

(a) MAP Detector : Recall that $\hat{s}_{MAP}(r) = \arg\max_s p_s f_N(r-s)$

This is true regardless of the pdf of the noise

Here, there are two possible values for s.

So we compare $\underset{0.3}{\underset{\uparrow}{p_1}} f_N(r - \underset{-3}{\underset{\uparrow}{s^{(1)}}})$ and $\underset{0.7}{\underset{\uparrow}{p_2}} f_N(r - \underset{3}{\underset{\uparrow}{s^{(2)}}})$



From the graph, it is clear that

1. $R < s^{(1)}$ is impossible. So, the detector can do anything in this region without affecting its performance.

2. When $s^{(1)} < r < s^{(2)}$, $p_1 f_N(r - s^{(1)}) > p_2 f_N(r - s^{(2)})$.
   So, in this region, $\hat{s}_{MAP}(r) = s^{(1)}$.

3. When $r > s^{(2)}$, $p_1 f_N(r - s^{(1)}) < p_2 f_N(r - s^{(2)})$.
   So, in this region, $\hat{s}_{MAP}(r) = s^{(2)}$.

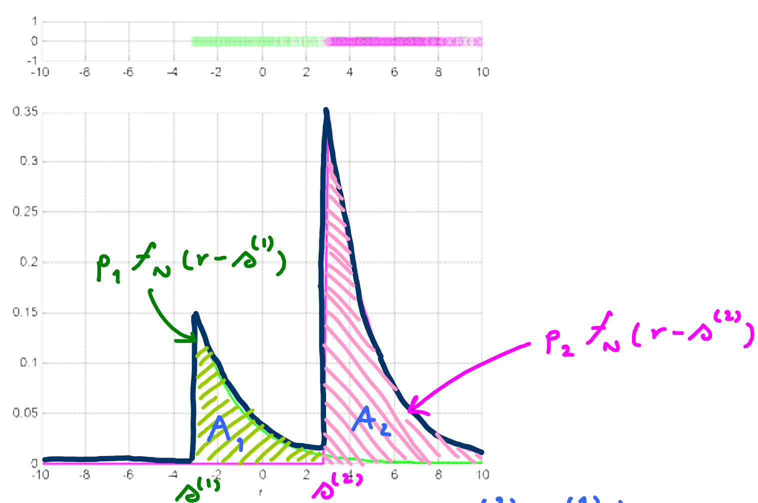
Conclusion:

$$\hat{s}_{MAP}(r) = \begin{cases} s^{(1)}, & s^{(1)} < r < s^{(2)}, \\ s^{(2)}, & r > s^{(2)}, \\ \text{anything}, & \text{otherwise} \end{cases} \underset{\text{simplification}}{=} \begin{cases} s^{(1)}, & r < s^{(2)} \\ s^{(2)}, & r \geq s^{(2)} \end{cases}$$

$\tau^* = s^{(2)} = 3$

$$= \begin{cases} -3, & r < 3 \\ 3, & r \geq 3 \end{cases}$$

(b) Recall that for MAP detector, $P(C)$ = area under the "max." plot.



Area $A_1 = p_1 - p_1 e^{-\lambda(s^{(2)} - s^{(1)})}$

Area $A_2 = p_2$

So, $P(C) = 1 - p_1 e^{-\lambda(s^{(2)} - s^{(1)})}$ and

$$P(\mathcal{E}) = 1 - P(C) = p_1 e^{-\lambda(s^{(2)} - s^{(1)})} = 0.3\, e^{-\frac{1}{2}(3 - (-3))}$$

$$= 0.3\, e^{-3} \approx 0.0149$$