Q1 GSOP for Complex-Valued Vectors

Monday, July 08, 2013 1:48 PM

(a) $\vec{u}^{(1)} = \vec{v}^{(1)} = \begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}$. $\quad \|\vec{u}^{(1)}\|^2 = (1^2+1^2) + (1^2+(-1)^2) + 0 = 2+2 = 4$

$\|\vec{u}^{(1)}\| = 2$

$$\vec{e}^{(1)} = \frac{\vec{u}^{(1)}}{\|\vec{u}^{(1)}\|} = \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}$$

$$\vec{u}^{(2)} = \vec{v}^{(2)} - \underbrace{\langle \vec{v}^{(2)}, \vec{e}^{(1)} \rangle}_{j} \vec{e}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - (-j)\frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} j-1 \\ j+1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}$$

$\left(\frac{1}{2}\right)^* \left((1-j) - (1+j)\right)$ Note that the j are conjugated.

$= \frac{1}{2}(-2j) = -j$

$$\|\vec{u}^{(2)}\|^2 = \frac{1}{2^2}\left((1^2+1^2) + (1^2+(-1)^2) + 0\right) = \frac{1}{2^2}(4) = 1$$

$$\vec{e}^{(2)} = \frac{\vec{u}^{(2)}}{\|\vec{u}^{(2)}\|} = \vec{u}^{(2)} = \frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(3)} = \vec{v}^{(3)} - \langle \vec{v}^{(3)}, \vec{e}^{(1)} \rangle \vec{e}^{(1)} - \langle \vec{v}^{(3)}, \vec{e}^{(2)} \rangle \vec{e}^{(2)}$$

with $\vec{v}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\vec{e}^{(1)} = \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}$, $\vec{e}^{(2)} = \frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}$

$\langle \vec{v}^{(3)}, \vec{e}^{(1)} \rangle = \left(\frac{1}{2}\right)^* \left((1-j) + (1+j) + 0\right) = \frac{1}{2}(2) = 1$

$\langle \vec{v}^{(3)}, \vec{e}^{(2)} \rangle = \left(\frac{1}{2}\right)^* \left(-j+1 - j - 1\right) = \frac{1}{2}(-2j) = -j$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix} + j\frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 2 & -1-j & +\,-1+j \\ 2 & -1+j & +\,-1-j \\ -2 & +0 & +0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\|\vec{u}^{(3)}\| = \sqrt{0^2+0^2+(-1)^2} = 1$$

$$\vec{e}^{(3)} = \frac{\vec{u}^{(3)}}{\|\vec{u}^{(3)}\|} = \vec{u}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

$$\vec{u}^{(4)} = \vec{v}^{(4)} - \langle \vec{v}^{(4)}, \vec{e}^{(1)} \rangle \vec{e}^{(1)} - \langle \vec{v}^{(4)}, \vec{e}^{(2)} \rangle \vec{e}^{(2)} - \langle \vec{v}^{(4)}, \vec{e}^{(3)} \rangle \vec{e}^{(3)}$$

with $\vec{v}^{(4)} = \begin{pmatrix} -1 \\ -1 \\ -j \end{pmatrix}$, $\vec{e}^{(1)} = \frac{1}{2}\begin{pmatrix} 1+j \\ 1-j \\ 0 \end{pmatrix}$, $\vec{e}^{(2)} = \frac{1}{2}\begin{pmatrix} j+1 \\ j-1 \\ 0 \end{pmatrix}$, $\vec{e}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\langle \vec{v}^{(4)}, \vec{e}^{(1)} \rangle = \left(\frac{1}{2}\right)^* \left(-(1-j) - (1+j)\right) = \frac{1}{2}\left(-1+j-1-j\right) = \frac{-2}{2} = -1$

$\langle \vec{v}^{(4)}, \vec{e}^{(2)} \rangle = \left(\frac{1}{2}\right)^* \left(-(-j+1) - (-j-1)\right) = \frac{1}{2}\left(j-1+j+1\right) = \frac{1}{2}(2j) = j$

$\langle \vec{v}^{(4)}, \vec{e}^{(3)} \rangle = (-j)(-1) = j$

$$= \frac{1}{2}\begin{pmatrix} -2 & +1+j & -(-1+j) & +0 \\ -2 & +1-j & -(-1-j) & +0 \\ -2j & +0 & -0 & +2j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow \text{so, no } \vec{u}^{(4)} \text{ and hence no } \vec{e}^{(4)}.$$

$$= \frac{1}{2}\begin{pmatrix} -2 & +1+j & -(-1+j) & +0 \\ -2 & +1-j & -(-1-j) & +0 \\ -2j & +0 & -\ 0 & +2j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow \text{so, no } \vec{u}^{(4)} \text{ and hence no } \vec{e}^{(4)}.$$

(b) From the procedure above, we get

$$\vec{u}^{(1)} = \vec{v}^{(1)} \quad \Rightarrow \vec{v}^{(1)} = \vec{u}^{(1)} = \|\vec{u}^{(1)}\|\vec{e}^{(1)} = 2\vec{e}^{(1)} = E\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(2)} = \vec{v}^{(2)} - (-j)\,\vec{e}^{(1)} \quad \Rightarrow \vec{v}^{(2)} = (-j)\,\vec{e}^{(1)} + \|\vec{u}^{(2)}\|\vec{e}^{(2)} = (-j)\,\vec{e}^{(1)} + 1\,\vec{e}^{(2)} = E\begin{pmatrix} -j \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}^{(3)} = \vec{v}^{(3)} - 1\,\vec{e}^{(1)} - (-j)\,\vec{e}^{(2)} \quad \Rightarrow \vec{v}^{(3)} = (1)\,\vec{e}^{(1)} + (-j)\,\vec{e}^{(2)} + \|\vec{u}^{(3)}\|\vec{e}^{(3)}$$

$$= (1)\,\vec{e}^{(1)} + (-j)\,\vec{e}^{(2)} + (1)\,\vec{e}^{(3)} = E\begin{pmatrix} 1 \\ -j \\ 1 \end{pmatrix}$$

$$\underbrace{\vec{u}^{(4)}}_{0} = \vec{v}^{(4)} - (-1)\,\vec{e}^{(1)} - (j)\,\vec{e}^{(2)} - (j)\,\vec{e}^{(3)} \quad \Rightarrow \vec{v}^{(4)} = (-1)\,\vec{e}^{(1)} + (j)\,\vec{e}^{(2)} + (j)\,\vec{e}^{(3)} = E\begin{pmatrix} -1 \\ j \\ j \end{pmatrix}$$

$$\text{So,} \quad V = \left[\vec{v}^{(1)}\ \ \vec{v}^{(2)}\ \ \vec{v}^{(3)}\ \ \vec{v}^{(4)}\right] = E\underbrace{\begin{bmatrix} 2 & -j & 1 & -1 \\ 0 & 1 & -j & j \\ 0 & 0 & 1 & j \end{bmatrix}}_{C}.$$

Q2 Signal Space and Constellation

Monday, July 08, 2013 10:39 AM

(a)

$$\mathcal{E}_1 = \mathcal{E}_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b .$$

$$\mathcal{E}_2 = \mathcal{E}_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b .$$

$$\equiv \mathcal{E}$$

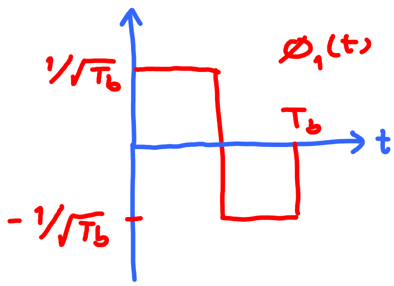
(b)

$$u_1(t) = s_1(t).$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \frac{1}{\sqrt{\mathcal{E}}} s_1(t)$$

$$\frac{V}{\sqrt{\mathcal{E}}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$

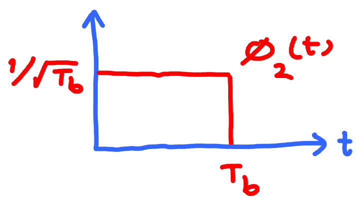
$$= \begin{cases} 1/\sqrt{T_b}, & 0 \le t \le \frac{T_b}{2} \\ -1/\sqrt{T_b}, & \frac{T_b}{2} < t < T_b \\ 0, & \text{otherwise} \end{cases}$$



$$u_2(t) = s_2(t) - \mathrm{proj}_{u_1} s_2 = s_2(t) - \frac{\overbrace{\langle s_2, s_1 \rangle}^{0}}{\langle s_1, s_1 \rangle} s_1(t) = s_2(t)$$

$$\phi_2(t) = \frac{u_2(t)}{\sqrt{\mathcal{E}_{u_2}}} \underset{u_2 = s_2}{=} \frac{s_2(t)}{\sqrt{\mathcal{E}_2}} = \frac{1}{\sqrt{\mathcal{E}}} s_2(t)$$

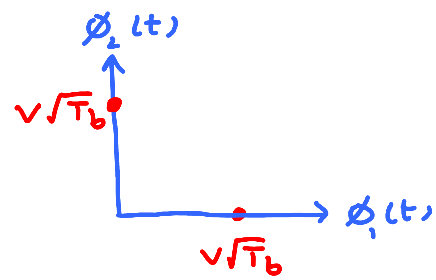
$$= \begin{cases} 1/\sqrt{T_b}, & 0 \le t \le T_b, \\ 0, & \text{otherwise.} \end{cases}$$



(c)

$$s_1(t) = \sqrt{\mathcal{E}}\, \phi_1(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} \sqrt{\mathcal{E}} \\ 0 \end{pmatrix} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix}.$$

$$s_2(t) = \sqrt{\mathcal{E}}\, \phi_2(t) \Rightarrow \vec{s}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{\mathcal{E}} \end{pmatrix} = \begin{pmatrix} 0 \\ V\sqrt{T_b} \end{pmatrix}.$$



Q3 Signal Space and Constellation

Monday, July 08, 2013 11:17 AM

(a)

$$\mathcal{E}_1 = \mathcal{E}_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

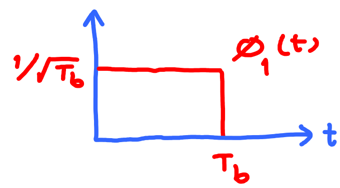
$$\mathcal{E}_2 = \mathcal{E}_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$\equiv \mathcal{E}$$

(b)

$$u_1(t) = s_1(t).$$

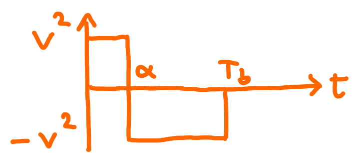
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} = \frac{1}{\sqrt{\mathcal{E}}} s_1(t) = \begin{cases} 1/\sqrt{T_b}, & 0 < t < T_b \\ 0, & \text{otherwise.} \end{cases}$$



$$\frac{V}{\sqrt{\mathcal{E}}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$

$$u_2(t) = s_2(t) - \text{proj}_{u_1} s_2 = s_2(t) - \frac{\langle s_2, s_1 \rangle}{\langle s_1, s_1 \rangle} s_1(t)$$

$$\langle s_2, s_1 \rangle = \int$$



$$dt$$

$$= \alpha V^2 - (T_b - \alpha) V^2 = 2\alpha V^2 - T_b V^2$$

$$u_2(t) = s_2(t) - \frac{2\alpha \cancel{V^2} - T_b \cancel{V^2}}{\cancel{V^2} T_b} s_1(t) = s_2(t) - \left(\frac{2\alpha}{T_b} - 1 \right) s_1(t)$$

$$= s_2(t) + \underbrace{\left(1 - \frac{2\alpha}{T_b}\right)} s_1(t)$$

when $\alpha = \frac{T_b}{2}$, this term $= 0$ and $u_2(t) = s_2(t)$.

when $\alpha < \frac{T_b}{2}$, this term is > 0. So, $s_2(t)$ will be shifted up.

when $\alpha > \frac{T_b}{2}$, this term is < 0. So, $s_2(t)$ will be shifted down.

$$\begin{cases} V + \left(1 - \frac{2\alpha}{T_b}\right) V & \text{for } 0 < t < \alpha, \end{cases}$$

$$= \begin{cases} V + \left(1-\frac{2\alpha}{T_b}\right)V & \text{for } 0<t<\alpha, \\ -V + \left(1-\frac{2\alpha}{T_b}\right)V & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left(2-\frac{2\alpha}{T_b}\right)V & \text{for } 0<t<\alpha, \\ -\frac{2\alpha}{T_b}V & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases} = 2V \times \begin{cases} 1-\frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

Note: The shift amount is just enough to make $\int u_2 = 0$

$$\left[\left(1-\frac{\alpha}{T_b}\right)\alpha + \left(-\frac{\alpha}{T_b}\right)(T_b-\alpha) = \alpha - \frac{\alpha^2}{T_b} - \alpha + \frac{\alpha^2}{T_b} = 0.\right]$$

$$\mathcal{E}_{u_2} = 4V^2\left(\underbrace{\left(1-\frac{\alpha}{T_b}\right)^2}_{1-\frac{2\alpha}{T_b}+\frac{\alpha^2}{T_b^2}}\alpha + \left(-\frac{\alpha}{T_b}\right)^2(T_b-\alpha)\right) = 4V^2\left(\alpha - \frac{\cancel{2}\alpha^2}{T_b} + \cancel{\frac{\alpha^3}{T_b^2}} + \cancel{\frac{\alpha^2}{T_b}} - \cancel{\frac{\alpha^3}{T_b^2}}\right)$$

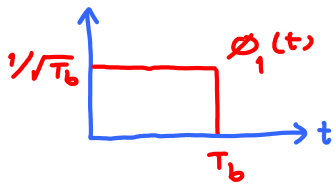
$$= 4V^2\alpha\left(1-\frac{\alpha}{T_b}\right) \quad \Rightarrow \sqrt{\mathcal{E}_{u_2}} = 2V\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}$$

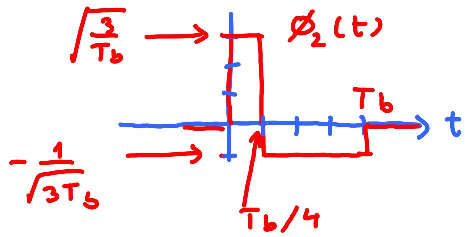
$$\phi_2(t) = \frac{u_2(t)}{\sqrt{\mathcal{E}_{u_2}}} = \frac{\cancel{2V}}{\cancel{2V}\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}} \begin{cases} 1-\frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \frac{1}{\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}} \times \begin{cases} 1-\frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

(c) When $\alpha = \frac{1}{4}T_b$, $\;\; = \dfrac{1}{\sqrt{\frac{T_b}{4}\left(1-\frac{1}{4}\right)}} = \dfrac{4}{\sqrt{3T_b}}$

$$\phi_2(t) = \frac{4}{\sqrt{3T_b}} \times \begin{cases} 3/4 & \text{for } 0<t<\frac{T_b}{4} \\ -1/4 & \text{for } \frac{T_b}{4}<t<T_b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \sqrt{3/T_b}, & \text{for } 0<t<\frac{T_b}{4}, \\ -\frac{1}{\sqrt{3T_b}}, & \text{for } \frac{T_b}{4}<t<T_b, \\ 0, & \text{otherwise.} \end{cases}$$





(d)

$$s_1(t) = \sqrt{\mathcal{E}_1}\,\phi_1(t) = \sqrt{\mathcal{E}}\,\phi_1(t) = V\sqrt{T_b}\,\phi_1(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\mathcal{E}} \\ 0 \end{pmatrix}.$$

$$s_2(t) = u_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right) s_1 = \sqrt{\mathcal{E}_{u_2}}\,\phi_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right)\sqrt{\mathcal{E}}\,\phi_1(t)$$

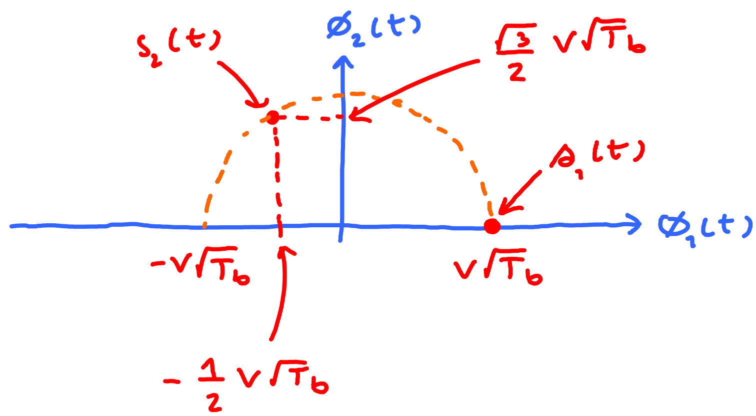
$$= 2V\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}\,\phi_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right)V\sqrt{T_b}\,\phi_1(t).$$

$$\Rightarrow \vec{s}^{(2)} = \begin{pmatrix} -\left(1 - 2\frac{\alpha}{T_b}\right) \\ 2\sqrt{\frac{\alpha}{T_b}\left(1 - \frac{\alpha}{T_b}\right)} \end{pmatrix} \times V\sqrt{T_b} = \sqrt{\mathcal{E}}\begin{pmatrix} 2r - 1 \\ 2\sqrt{r(1-r)} \end{pmatrix} \quad \text{where } r = \frac{\alpha}{T_b}.$$

(e)

When $\alpha = \frac{T_b}{4}$, $s_2(t) = 2V\sqrt{\frac{T_b}{4}\left(\frac{3}{4}\right)}\,\phi_2(t) - \left(1 - \frac{1}{2}\right)V\sqrt{T_b}\,\phi_1(t)$

$$= \frac{\sqrt{3}}{2}V\sqrt{T_b}\,\phi_2(t) - \frac{1}{2}V\sqrt{T_b}\,\phi_1(t)$$



(f)

Note that $(2r-1)^2 + \left(2\sqrt{r(1-r)}\right)^2 = 4r^2 - 4r + 1 + 4r - 4r^2 = 1.$

$$r = \frac{\alpha}{T_b} = \frac{k}{10}$$