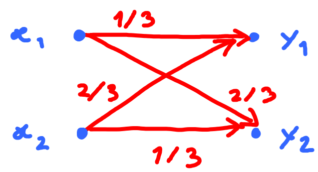
Q1

Wednesday, October 02, 2013 10:51 AM

(a) (i) The problem does not explicitly specify $\mathcal{X}$ and $\mathcal{Y}$. However, from the size of the Q matrix, we know that $|\mathcal{X}| = |\mathcal{Y}| = 2$.
So, we will set $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$.



(ii) Note that this is a BSC with $p = \frac{2}{3}$. We know that the capacity of BSC is

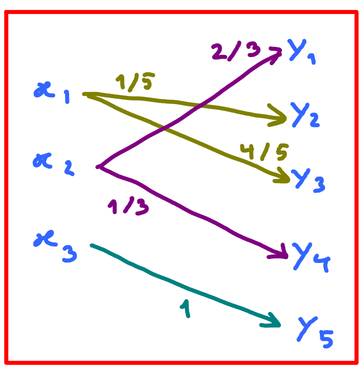
$$C = 1 - H(p) = 1 - H\left(\frac{2}{3}\right) = 1 - 0.9183 \approx 0.0817$$

Alternatively, this channel is weakly symmetric. Therefore,

$$C = \log_2 |S_Y| - H(\underline{r}) = \log_2 2 - H\left(\left[\frac{1}{3} \ \frac{2}{3}\right]\right)$$

$$= 1 - 0.9183 \approx 0.0817$$

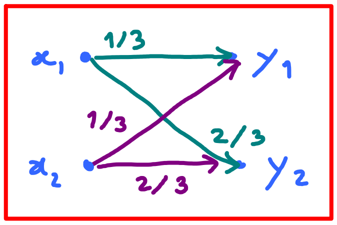
(b) (i) Let $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2, y_3, y_4, y_5\}$.



(ii) Note that this is a noisy channel with non overlapping outputs.
(Only one non-zero element in each column of Q.)

So, $C = \log_2 |S_X| = \log_2 3 \approx 1.5850$.

(c) (i) As in (a), we let $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$



(ii) The rows of Q are the same.

So, $Q(y|x)$ does not depend on x.

can "pull" $Q(y|x)$ out of the summation

Therefore, $q(y) = \sum_x p(x) Q(y|x) = Q(y|x) \sum_x p(x) = Q(y|x)$,

which implies $X \perp\!\!\!\perp Y$.

Therefore, $I(X;Y) = 0$ for any input distribution.

Hence, $C = 0$.

Q2

Monday, October 27, 2014 4:51 PM

(a) There are $\underset{=|\mathcal{X}|}{3}$ possible channel inputs and $\underset{=|\mathcal{Y}|}{4}$ possible channel outputs.
Therefore, the Q matrix must be 3×4.

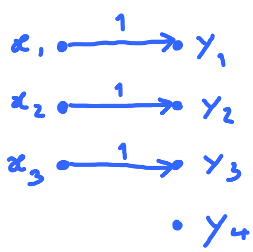
(b) We know that $C \le \min\{\underbrace{\log_2|\mathcal{X}|}_{\log_2 3}, \underbrace{\log_2|\mathcal{Y}|}_{\log_2 4}\} = \log_2 3 \approx 1.5850$

However $1.6 > 1.5850$. Therefore $C = 1.6$ bpcu is impossible.

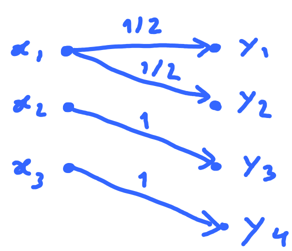
(c) Note that $\log_2 3 = \log_2|\mathcal{X}|$. Therefore,

let's try to construct a noiseless channel
or a noisy channel with non-overlapping output

Examples of answers:



or



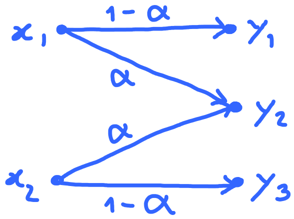
The correspond Q for the examples above are

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ or } Q = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ respectively.}$$

Q3a

Monday, October 27, 2014 4:59 PM

The given DMC belong to a family of channels called Binary Erasure Channel (BEC). The general form of the channel diagram and the corresponding Q matrix are shown below:

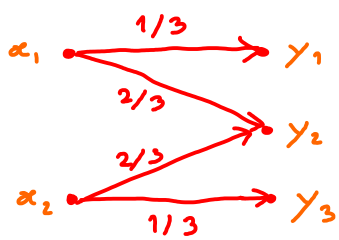


$$Q = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & \alpha & 1-\alpha \end{bmatrix}$$

Usually, $x_1 = y_1 = 0$, $x_2 = y_3 = 1$, $y_2 = e$

the case where the bit is "erased"

Here, we have $\alpha = 2/3$. So, the channel diagram is



(b) For general α,

$$Q = \begin{array}{c|ccc} x \backslash y & 0 & e & 1 \\ \hline 0 & 1-\alpha & \alpha & 0 \\ 1 & 0 & \alpha & 1-\alpha \end{array} \quad \begin{array}{l} \rightarrow H(Y|X=0) = H(\alpha) \\ \rightarrow H(Y|X=1) = H(\alpha) \end{array} \Bigg\} \Rightarrow H(Y|X) = H(\alpha)$$

Let $\underline{p} = [p_0 \;\; 1-p_0]$. Then,

Write these as $\bar{\alpha}$

$$\underline{q} = \underline{p}Q = [p_0 \;\; 1-p_0] \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & \alpha & 1-\alpha \end{bmatrix} = [p_0 \overbrace{(1-\alpha)} \;\; \alpha \;\; \underbrace{(1-p_0)}\overbrace{(1-\alpha)}]$$

Write this as $\bar{p_0}$

$$= [p_0\bar{\alpha} \;\; \alpha \;\; \bar{p}_0\bar{\alpha}]$$

Therefore, $H(Y) = \underbrace{-p_0\bar{\alpha}\log_2 p_0\bar{\alpha} - \alpha\log_2\alpha - \bar{p}_0\bar{\alpha}\log_2\bar{p}_0\bar{\alpha}}$

and

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(\alpha).$$

Some useful facts: For any positive function $f(x)$,

$$\frac{d}{dx} f(x)\ln f(x) = f'(x)\ln f(x) + f(x)\frac{1}{f(x)}f'(x)$$

$$= f'(x)\left(1+\ln f(x)\right) = f'(x)\ln\left(e f(x)\right)$$

$$\Big\downarrow \times \frac{1}{\ln 2}$$

$$\frac{d}{dx} f(x)\log_2 f(x) \quad = f'(x)\log_2 e f(x)$$

$$\frac{d}{dp_0} I(X;Y) = -\cancel{\bar{\alpha}}\log_2(\cancel{e}p_0\cancel{\bar{\alpha}}) - (-\cancel{\bar{\alpha}})\log_2(\cancel{e}\bar{p}_0\cancel{\bar{\alpha}}) = 0$$

$$p_0 = \bar{p}_0$$
$$p_0 = 1-p_0$$
$$p_0 = \frac{1}{2}$$

One may say that the uniform pmf is "obvious" from the "symmetry" in the transition probabilities from the inputs.

With $p_0 = \frac{1}{2}$, we have

$$C = I(X;Y) = -\frac{\bar{\alpha}}{2}\log_2\frac{\bar{\alpha}}{2} - \alpha\cancel{\log_2\alpha} - \frac{\bar{\alpha}}{2}\log_2\frac{\bar{\alpha}}{2} - \left(-\cancel{\alpha\log_2\alpha} - \bar{\alpha}\log_2\bar{\alpha}\right)$$

$$= -\bar{\alpha}\log_2\frac{\bar{\alpha}}{2} + \bar{\alpha}\log_2\bar{\alpha} = \bar{\alpha}\log_2\frac{\bar{\alpha}}{\bar{\alpha}/2} = \bar{\alpha} = 1-\alpha$$

When $\alpha = \frac{2}{3}$, $\quad C = 1-\frac{2}{3} = \frac{1}{3}$ bpcu.

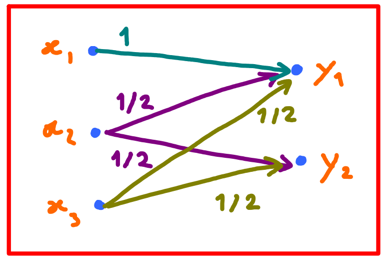
The capacity-achieving $\bar{p}$ is $\bar{p} = \left[\frac{1}{2} \quad \frac{1}{2}\right]$.

Q3b

Tuesday, October 28, 2014 10:08 AM

(i)

$$Q = \begin{array}{c|cc} x \backslash y & y_1 & y_2 \\ \hline x_1 & 1 & 0 \\ x_2 & 1/2 & 1/2 \\ x_3 & 1/2 & 1/2 \end{array} \Rightarrow$$



(ii) As hinted, we let $\underline{p} = [p_1\ p_2\ p_3]$.

Then,

$$\underline{q} = \underline{p}\,Q = [p_1\ \ p_2\ \ p_3]\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$= \left[p_1 + \tfrac{1}{2}p_2 + \tfrac{1}{2}p_3 \quad 0 + \tfrac{1}{2}p_2 + \tfrac{1}{2}p_3\right] = \left[p_1 + \frac{p_2+p_3}{2} \quad \frac{p_2+p_3}{2}\right]$$

Next, from each row of the Q matrix, we can find

$H(Y|X=x_1) = H([1\ 0]) = 0$,

$H(Y|X=x_2) = H([1/2\ 1/2]) = \log_2 2 = 1$, and

$H(Y|X=x_3) =$ " " $= 1$.

Therefore,

$$H(Y|X) = \sum_x p(x) H(Y|x) = p_1 \times 0 + p_2 + p_3 = p_2 + p_3.$$

Note that $\underline{q}$ and $H(Y|X)$ do not depend directly on the values of p_2 and p_3. They use p_2 and p_3 via the sum p_2+p_3.

Hence, $H(Y)$ (which is calculated from $\underline{q}$) and $I(X;Y) = H(Y) - H(Y|X)$ also do not depend directly on p_2 and p_3. Only the sum p_2+p_3 is involved.

Note that $p_1+p_2+p_3 = 1$. So, $p_2+p_3 = 1-p_1$.

Hence,

$$\underline{q} = \left[p_1 + \frac{p_2+p_3}{2} \quad \frac{p_2+p_3}{2}\right] = \left[p_1 + \frac{1-p_1}{2} \quad \frac{1-p_1}{2}\right] = \left[\frac{1+p_1}{2} \quad \frac{1-p_1}{2}\right]$$

$$H(Y|X) = p_2 + p_3 = 1 - p_1$$

So, $I(X;Y) = H(Y) - H(Y|X) = -\frac{1+p_1}{2}\log_2\frac{1+p_1}{2} - \frac{1-p_1}{2}\log_2\frac{1-p_1}{2} - (1-p_1)$

$$\frac{d}{dp_1} I(X;Y) = -\frac{1}{2}\log_2 e\,\frac{1+p_1}{2} - \left(-\frac{1}{2}\right)\log_2 e\,\frac{1-p_1}{2} + 1 = 0$$

$$\frac{1}{2}\log_2\frac{1-p_1}{1+p_1} + 1 = 0$$

$$\frac{1+p_1}{1-p_1} = 2^2 = 4$$

$$1+p_1 = 4-4p_1$$

$$5p_1 = 3$$

$$p_1 = \frac{3}{5}$$

With $p_1 = \frac{3}{5}$,

$$\underline{q} = \begin{bmatrix}\frac{1+\frac{3}{5}}{2} & \frac{1-\frac{3}{5}}{2}\end{bmatrix} = \begin{bmatrix}\frac{4}{5} & \frac{1}{5}\end{bmatrix}$$

So, $C = H\left(\begin{bmatrix}\frac{4}{5} & \frac{1}{5}\end{bmatrix}\right) - \left(1-\frac{3}{5}\right) \approx 0.7219 - \frac{2}{5} = 0.3219$ bpcu

The pmf $\underline{p}$ of X that achieves this capacity value is

$$\underline{p} = \begin{bmatrix}\frac{3}{5} & p_2 & p_3\end{bmatrix} \quad \text{where} \quad p_2 + p_3 = \frac{2}{5}$$

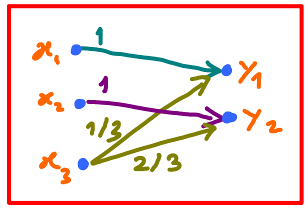
Ex. $p_2 = p_3 = \frac{1}{5}$

Q4

Monday, October 27, 2014 4:30 PM

(a)

$$Q = \begin{array}{c|cc} x \backslash y & y_1 & y_2 \\ \hline x_1 & 1 & 0 \\ x_2 & 0 & 1 \\ x_3 & 1/3 & 2/3 \end{array} \Rightarrow$$



(b)

Note that $|\mathcal{Y}| = 2$. So, $I(X;Y) \le H(Y) \le \log_2 |\mathcal{Y}| = \log_2 2 = 1.$

Note also that without the last row, this Q corresponds to a noiseless channel with capacity $\log_2 |\mathcal{X}| = \log_2 2 = 1$ which is achieved by uniform pmf on the input. ($\underline{p} = [\frac{1}{2} \ \frac{1}{2}]$.)

Now, when the last row of Q is included, we may choose not to use it by using $\underline{p} = [\frac{1}{2} \ \frac{1}{2} \ 0]$. This gives

$$\underline{q} = [\tfrac{1}{2} \ \ \tfrac{1}{2} \ \ 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/3 & 2/3 \end{bmatrix} = [\tfrac{1}{2} \ \ \tfrac{1}{2}]$$

So, $H(Y) = H([\frac{1}{2} \ \frac{1}{2}]) = 1$ and

$$H(Y|X) = \frac{1}{2} \times \underbrace{H([1 \ \ 0])}_{0} + \frac{1}{2} \times \underbrace{H([0 \ \ 1])}_{0} + 0 \times H([\tfrac{1}{3} \ \ \tfrac{2}{3}]) = 0.$$

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 1 - 0 = 1$ which is the same as the bound above.

Because $I(X;Y)$ can not exceed the bound, we know that this $\underline{p} = [\frac{1}{2} \ \frac{1}{2} \ 0]$ achieved the maximum $I(X;Y)$.

Therefore, $C = 1$ bpcu and corresponding $\underline{p} = [\frac{1}{2} \ \frac{1}{2} \ 0]$

Q5

Friday, October 31, 2014 9:53 AM

```
close all; clear all;
```

```
>> Capacity_HW_blahut
```

```
%% 1.a
Q = [1/3 2/3; 2/3 1/3];
[ps C] = capacity_blahut(Q)
```

ps =
[0.5000 0.5000] = p
C =
0.0817 bpcu

```
%% 1.b
Q = [0 1/5 4/5 0 0; 2/3 0 0 1/3 0; 0 0 0 0 1];
[ps C] = capacity_blahut(Q)
```

ps =
[0.3333 0.3333 0.3333] = p
C =
1.5850 bpcu

```
%% 1.c
Q = [1/3 2/3; 1/3 2/3];
[ps C] = capacity_blahut(Q)
```

ps =
[0.5000 0.5000] = p
C =
0 bpcu

```
%% 2.c
Q = [1 0 0 0; 0 1 0 0; 0 0 1 0];
[ps C] = capacity_blahut(Q)
```

ps =
[0.3333 0.3333 0.3333] = p
C =
1.5850 bpcu

```
%% 3.a
Q = [1/3 2/3 0; 0 2/3 1/3];
[ps C] = capacity_blahut(Q)
```

ps =
[0.5000 0.5000] = p
C =
0.3333

```
%% 3.b
Q = [1 0; 1/2 1/2; 1/2 1/2];
[ps C] = capacity_blahut(Q)
```

ps =
[0.6000 $\underbrace{0.2000 \quad 0.2000}_{\text{one of the answers}}$] = p
C =
0.3219 bpcu

```
%% 4
Q = [1 0; 0 1; 1/3 2/3];
[ps C] = capacity_blahut(Q)
```

ps =
[0.5000 0.5000 0.0000] = p
C =
1.0000 bpcu