ECS 452: Digital Communication Systems **2014/1**

HW 3 — Not Due

Lecturer: Asst. Prof. Dr. Prapun Suksompong

Problem 1. ***Optimal code lengths that require one bit above entropy***: The source coding theorem says that the Huffman code for a random variable X has an expected length strictly less than $H(X) + 1$. Give an example of a random variable for which the expected length of the Huffman code (without any source extension) is very close to $H(X) + 1$.

Problem 2. Consider a repetition code with a code rate of $1/5$. Assume that the code is used with a BSC with a crossover probability $p = 0.4$.

(a) Find the ML detector and its error probability.

(b) Suppose the info-bit S is generated with

$$P[S = 0] = 1 - P[S = 1] = 0.4.$$

Find the MAP detector and its error probability.

(c) Assume the info-bit S is generated with

$$P[S = 0] = 1 - P[S = 1] = 0.45.$$

Suppose the receiver observes 01001.

(i) What is the probability that 0 was transmitted? (Do not forget that this is a conditional probability. The answer is not 0.45 because we have some extra information from the observed bits at the receiver.)

(ii) What is the probability that 1 was transmitted?

(iii) Given the observed 01001, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? Does your answer agree with the majority voting rule for decoding?

(d) Assume that the source produces source bit S with

$$P[S = 0] = 1 - P[S = 1] = p_0.$$

Suppose the receiver observes 01001.

(i) What is the probability that 0 was transmitted?

(ii) What is the probability that 1 was transmitted?

(iii) Given the observed 01001, which event is more likely, $S = 1$ was transmitted or $S = 0$ was transmitted? Your answer may depend on the value of p_0. Does your answer agree with the majority voting rule for decoding?

Problem 3. A channel encoder map blocks of two bits to five-bit (channel) codewords. The four possible codewords are 00000, 01000, 10001, and 11111. A codeword is transmitted over the BSC with crossover probability $p = 0.1$. Suppose the receiver observes 01001 at the output of the BSC.

(a) Assume that all four codewords are equally likely to be transmitted. Given the observed 01001 at the receiver, what is the most likely codeword that was transmitted?

(b) What is the minimum (Hamming) distance d_{min} among the codewords?

(c) Assume that the four codewords are not equally likely. Suppose 11111 is transmitted more frequently with probability 0.7. The other three codewords are transmitted with probability 0.1 each.

Given the observed 01001 at the receiver, what is the most likely codeword that was transmitted?

Problem 4. Consider random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} c\,(x+y)\,, & x \in \{1,3\} \text{ and } y \in \{2,4\}\,, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the following quantities.

(a) $H(X,Y)$

(b) $H(X)$

(c) $H(Y)$

(d) $H(X|Y)$

(e) $H(Y|X)$

(f) $I(X;Y)$

Problem 5. Consider a pair of random variables X and Y whose joint pmf is given by

$$p_{X,Y}(x,y) = \begin{cases} 1/15, & x = 3, y = 1, \\ 2/15, & x = 4, y = 1, \\ 4/15, & x = 3, y = 3, \\ \beta, & x = 4, y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent?

(b) Evaluate the following quantities.

(i) $H(X, Y)$

(ii) $H(X)$

(iii) $H(Y)$

(iv) $H(X|Y)$

(v) $H(Y|X)$

(vi) $I(X; Y)$