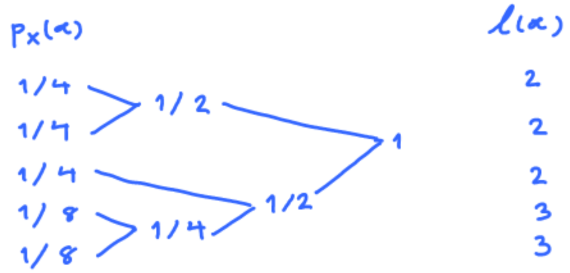
ECS 452: Quiz 1

Instructions

1. Separate into groups of no more than three persons.
2. Only one submission is needed for each group.
3. ***Write down all the steps*** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
4. **Do not panic.**

Name	ID

1. Consider a random variable X which has five possible values. Their probabilities are 1/4, 1/4, 1/4, 1/8, 1/8.
   a. Find the expected codeword length when Huffman coding is used without extension.



$$\mathbb{E}[\ell(X)] = 3\times 2\times \frac{1}{4} + 2\times 3\times \frac{1}{8}$$
$$= \frac{6}{4}+\frac{3}{4} = \frac{9}{4} = 2.25 \text{ bits/symbol}$$

   b. Find the entropy (per symbol) of this random variable.

$$H(X) = \sum_{x} p_X(x)\log_2 p_X(x) = -3\times\frac{1}{4}\log_2\frac{1}{4} - 2\times\frac{1}{8}\log_2\frac{1}{8}$$
$$= 3\times\frac{2}{4} + 2\times\frac{3}{8} = \frac{9}{4} = 2.25 \text{ bits/symbol}$$

2. No need to provide any explanation for this question.
   Consider a DMC whose samples of input and output are provided below

```
x: 1  1  1  0  1  0  1  0  1  1  1  1  1  1  1
y: 1  1  1  0  1  0  1  1  1  1  1  1  1  1  1
```

   Estimate the following quantities:

   a. $P[X = 0] \approx 3/15 = 1/5 = 0.2$

   b. $p(1) \equiv P[X=1] \approx 12/15 = 4/5 = 0.8$

   c. $p_Y(0) \equiv P[Y=0] \approx 2/15 \approx 0.133$

   d. $q(1) \equiv P[Y=1] \approx 13/15 \approx 0.867$

   e. $P[Y = 0 \mid X = 0] \approx \frac{2}{3} \approx 0.667$

   f. $p_{Y|X}(1|0) \equiv P[Y=1 \mid X=0] \approx \frac{1}{3} \approx 0.333$

   g. $Q(0|1) \equiv [Y=0 \mid X=1] \approx \frac{0}{15} = 0$

   h. $Q(1|1) \equiv P[Y=1 \mid X=1] \approx \frac{15}{15} = 1$

   i. Matrix Q $\approx$

$$\begin{array}{c|c} x \backslash y & \begin{matrix} 0 & \quad 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{bmatrix} 2/3 & 1/3 \\ 0 & 1 \end{bmatrix} \end{array}$$

   j. $P[X = 0, Y = 0] \approx \frac{2}{15}$ ← Note that this is the same as

$$P[Y=0 \mid X=0]\,P[X=0]$$
$$\frac{2}{3}\times\frac{1}{5} = \frac{2}{15}$$

ECS 452: Quiz 2 Solution

Instructions

1. Separate into groups of no more than three persons.
2. The group cannot be the same as your former group.
3. Only one submission is needed for each group.
4. ***Write down all the steps*** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
5. **Do not panic.**

Name	ID
Prapun	555

1. Consider a DMC whose $\mathcal{X}=\{1,2,3\}$, $\mathcal{Y}=\{1,2,3,4\}$, and $\mathbf{Q}=\begin{bmatrix}0.2 & 0.6 & 0.1 & 0.1\\ 0.1 & 0.7 & 0.1 & 0.1\\ 0.3 & 0.3 & 0.3 & 0.1\end{bmatrix}$.

   Suppose the prior probability vector is $\underline{\mathbf{p}}=[0.2 \quad 0.1 \quad 0.7]$.

   a. Find the joint pmf matrix $\mathbf{P}$.

   Multiply each row in the Q matrix by its corresponding p(x)

$$Q = \begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.2 & 0.6 & 0.1 & 0.1 \\ 2 & 0.1 & 0.7 & 0.1 & 0.1 \\ 3 & 0.3 & 0.3 & 0.3 & 0.1 \end{array} \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.1} \\ \xrightarrow{\times 0.7} \end{array} \begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.04 & 0.12 & 0.02 & 0.02 \\ 2 & 0.01 & 0.07 & 0.01 & 0.01 \\ 3 & 0.21 & 0.21 & 0.21 & 0.07 \end{array} = P$$

   b. Find the MAP detector and its error probability.

$$P = \begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.04 & 0.12 & 0.02 & 0.02 \\ 2 & 0.01 & 0.07 & 0.01 & 0.01 \\ 3 & 0.21 & 0.21 & 0.21 & 0.07 \end{array}$$

   For each column of the P matrix, select the max value.

   The corresponding x-value for the selected value in each column.

y	$\hat{x}_{MAP}(y)$
1	3
2	3
3	3
4	3

   So, $\hat{x}_{MAP}(y) \equiv 3$.

   $P(\mathcal{C}) = 0.21+0.21+0.21+0.07 = 0.7$

   $P(\mathcal{E}) = 1-P(\mathcal{C}) = 1-0.7 = 0.3$

   c. Find the ML detector and its error probability.

   For each column of the Q matrix, select the max value.

$$Q = \begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.2 & 0.6 & 0.1 & 0.1 \\ 2 & 0.1 & 0.7 & 0.1 & 0.1 \\ 3 & 0.3 & 0.3 & 0.3 & 0.1 \end{array} \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.1} \\ \xrightarrow{\times 0.7} \end{array} \begin{array}{c|cccc} x \backslash y & 1 & 2 & 3 & 4 \\ \hline 1 & 0.04 & 0.12 & 0.02 & 0.02 \\ 2 & 0.01 & 0.07 & 0.01 & 0.01 \\ 3 & 0.21 & 0.21 & 0.21 & 0.07 \end{array} = P$$

   Select the same elements as in the Q matrix.

   The corresponding x-value for the selected value in each column

y	$\hat{x}_{ML}(y)$
1	3
2	2
3	3
4	1

   All values in the last column are the same. So, we can use any of them.

   $P(\mathcal{C}) = 0.21+0.07+0.21+0.02 = 0.51$

   $P(\mathcal{E}) = 1-0.51 = 0.49$

   Alternative answer

y	$\hat{x}_{ML}(y)$
1	3
2	2
3	3
4	2

   $P(\mathcal{C}) = 0.21+0.07+0.21+0.01 = 0.50$

   $P(\mathcal{E}) = 1-0.5 = 0.5$

   Another alternative answer

y	$\hat{x}_{ML}(y)$
1	3
2	2
3	3
4	3

   $P(\mathcal{C}) = 0.21+0.07+0.21+0.07 = 0.56$

   $P(\mathcal{E}) = 1-0.56 = 0.44$

ECS 452: Quiz 3 Solution

Instructions

1. Separate into groups of no more than three persons.
2. The group cannot be the same as your former groups.
3. Only one submission is needed for each group.
4. ***Write down all the steps*** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
5. **Do not panic.**

Name	ID
Prapun	555

1. Consider two random variables X and Y whose joint pmf matrix is given by $\mathbf{P} = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$. Find $I(X;Y)$.

We use the formula $I(X;Y) = H(X) + H(Y) - H(X,Y)$.

$H(X,Y)$ can be found directly from the elements in the P matrix:

$$H(X,Y) = -\frac{1}{2}\log_2\frac{1}{2} - 3\times\frac{1}{6}\log_2\frac{1}{6} = \frac{1}{2}\left(-\log_2\frac{1}{2} - \log_2\frac{1}{6}\right) = \frac{1}{2}\log_2 12 \approx 1.7925$$

$H(X)$ and $H(Y)$ can be found by first finding $p(x)$ and $q(y)$ from the Q matrix:

$$P = \begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix} \begin{matrix} \to \frac{2}{3} \\ \to \frac{1}{3} \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ 2/3 & 1/3 \end{matrix}$$

$H(X) = -\frac{2}{3}\log_2\frac{2}{3} - \frac{1}{3}\log_2\frac{1}{3} = 0.9183$

$H(Y) = 0.9183$

$q(y)$ has same probability values as $p(x)$

So, $I(X;Y) \approx 2\times 0.9183 - 1.7925$
$= 0.0441$

2. Consider two random variables X and Y whose $\underline{\mathbf{p}} = \left[\frac{1}{3}, \frac{2}{3}\right]$ and $\mathbf{Q} = \begin{bmatrix} 1/6 & 5/6 \\ 2/3 & 1/3 \end{bmatrix}$. Find $I(X;Y)$.

$$\begin{matrix} \times 1/3 \\ \times 2/3 \end{matrix} \to \begin{bmatrix} \frac{1}{18} & \frac{5}{18} \\ \frac{4}{9} & \frac{2}{9} \end{bmatrix} = P$$

First, we find the P matrix.

Then, we follow the same steps as in question (1).

$$P = \begin{bmatrix} 1/18 & 5/18 \\ 4/9 & 2/9 \end{bmatrix} \begin{matrix} \to 6/18 = 1/3 \\ \to 6/9 = 2/3 \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ 9/18 & 9/18 \\ \shortparallel & \shortparallel \\ 1/2 & 1/2 \end{matrix}$$

(same probability values)

$H(X) \approx 0.9183$

$H(Y) = \log_2 2 = 1$

uniform

$$H(X,Y) = -\frac{1}{18}\log_2\frac{1}{18} - \frac{5}{18}\log_2\frac{5}{18} - \frac{4}{9}\log_2\frac{4}{9} - \frac{2}{9}\log_2\frac{2}{9} \approx 1.7472$$

$I(X;Y) \approx H(X) + H(Y) - H(X,Y) = 0.9183 + 1 - 1.7472 = 0.1711$

3. Consider two random variables X and Y whose $\mathbf{Q} = \begin{bmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{bmatrix}$. Find $I(X;Y)$.

Note that the two rows in Q are identical. This means $Q(y|x)$ does not depend on x. In other words, knowing the value of X does not change the (conditional) pmf of Y. Therefore, X and Y are independent which implies $I(X;Y) = 0$.

See next page for a more direct solution.

Remark: Normally, to calculate $I(X;Y)$ you will need both $\underline{p}$ and Q.
So, there must be something special about Q that allows you to get $I(X;Y)$ without $\underline{p}$.

Direct calculation:

$H(Y|x) = H([\frac{1}{6}\ \frac{5}{6}]) \approx 0.65$ for any x.

So, $H(Y|X) = \sum_x p(x) H(Y|x) \approx 0.65 \underbrace{\sum_x p(x)}_{1} \approx 0.65$.

$I(X;Y) = H(Y) - H(Y|X)$. So, we need $H(Y)$ which in turn need $q(y)$

Let's try $p(x) = \begin{cases} 1-p_0 & x=0 \\ p_0 & x=1 \\ 0, & \text{otherwise} \end{cases}$

Then,
$$\overset{\underline{p}}{[1-p \quad p]} \overset{Q}{\begin{bmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{bmatrix}} = \overset{\underline{q}}{[\frac{1}{6}\ \frac{5}{6}]} \Rightarrow H(Y) = H([\frac{1}{6}\ \frac{5}{6}]) = H(Y|X)$$

regardless of the value of p

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.