# ECS 452: Quiz 3 Solution

## Instructions

1. Separate into groups of no more than three persons.
2. The group cannot be the same as your former groups.
3. Only one submission is needed for each group.
4. ***Write down all the steps*** that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
5. **Do not panic.**

| Name | ID |
|---|---|
| Prapun | 555 |
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1. Consider two random variables *X* and *Y* whose joint pmf matrix is given by $\mathbf{P}=\begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}$. Find $I(X;Y)$.

We use the formula $I(X;Y)=H(X)+H(Y)-H(X,Y)$.

$H(X,Y)$ can be found directly from the elements in the P matrix:

$$H(X,Y) = -\frac{1}{2}\log_2\frac{1}{2} - 3\times\frac{1}{6}\log_2\frac{1}{6} = \frac{1}{2}\left(-\log_2\frac{1}{2}-\log_2\frac{1}{6}\right) = \frac{1}{2}\log_2 12 \approx 1.7925$$

$H(X)$ and $H(Y)$ can be found by first finding $p(x)$ and $q(y)$ from the Q matrix:

$$P=\begin{bmatrix} 1/2 & 1/6 \\ 1/6 & 1/6 \end{bmatrix} \begin{matrix} \rightarrow \frac{2}{3} \\ \rightarrow \frac{1}{3} \end{matrix}$$

$\downarrow$ $\frac{2}{3}$ $\quad\downarrow$ $\frac{1}{3}$

$H(X) = -\frac{2}{3}\log_2\frac{2}{3}-\frac{1}{3}\log_2\frac{1}{3} = 0.9183$

$H(Y) = 0.9183$

$q(y)$ has same probability values as $p(x)$

So, $I(X;Y) \approx 2\times 0.9183 - 1.7925$
$= 0.0441$

2. Consider two random variables *X* and *Y* whose $\underline{\mathbf{p}}=\begin{bmatrix} \frac{1}{3}, & \frac{2}{3} \end{bmatrix}$ and $\mathbf{Q}=\begin{bmatrix} 1/6 & 5/6 \\ 2/3 & 1/3 \end{bmatrix}$. Find $I(X;Y)$.

$$\begin{matrix} \times 1/3 \\ \times 2/3 \end{matrix} \begin{bmatrix} \frac{1}{18} & \frac{5}{18} \\ \frac{4}{9} & \frac{2}{9} \end{bmatrix} = P$$

First, we find the P matrix.

Then, we follow the same steps as in question (1).

$$P=\begin{bmatrix} 1/18 & 5/18 \\ 4/9 & 2/9 \end{bmatrix} \begin{matrix} \rightarrow 6/18 = 1/3 \\ \rightarrow 6/9 = 2/3 \end{matrix}$$

$\downarrow$ $9/18 = 1/2$ $\quad\downarrow$ $9/18 = 1/2$

(same probability values)

$H(X) \approx 0.9183$

$H(Y) = \log_2 2 = 1$

uniform

$$H(X,Y) = -\frac{1}{18}\log_2\frac{1}{18} - \frac{5}{18}\log_2\frac{5}{18} - \frac{4}{9}\log_2\frac{4}{9} - \frac{2}{9}\log_2\frac{2}{9} \approx 1.7472$$

$I(X;Y) \approx H(X)+H(Y)-H(X,Y) = 0.9183+1-1.7472 = 0.1711$

3. Consider two random variables *X* and *Y* whose $\mathbf{Q}=\begin{bmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{bmatrix}$. Find $I(X;Y)$.

Note that the two rows in Q are identical. This means $Q(y|x)$ does not depend on $x$. In other words, knowing the value of $X$ does not change the (conditional) pmf of $Y$. Therefore, $X$ and $Y$ are independent which implies $I(X;Y)=0$.

See next page for a more direct solution.

Remark: Normally, to calculate $I(X;Y)$ you will need both $\underline{p}$ and $Q$.
So, there must be something special about $Q$ that allows you to get $I(X;Y)$ without $\underline{p}$.

Direct calculation:

$H(Y|x) = H([\frac{1}{6}\ \frac{5}{6}]) \approx 0.65$ for any $x$.

So, $H(Y|X) = \sum_x p(x) H(Y|x) \approx 0.65 \underbrace{\sum_x p(x)}_{1} \approx 0.65$.

$I(X;Y) = H(Y) - H(Y|X)$. So, we need $H(Y)$ which in turn need $q(y)$

Let's try $p(x) = \begin{cases} 1-p, & x=0 \\ p, & x=1 \\ 0, & \text{otherwise} \end{cases}$

Then,

$$\overset{\underline{p}}{[1-p \quad p]} \overset{Q}{\begin{bmatrix} 1/6 & 5/6 \\ 1/6 & 5/6 \end{bmatrix}} = \overset{\underline{q}}{[\frac{1}{6}\ \frac{5}{6}]} \Rightarrow H(Y) = H([\frac{1}{6}\ \frac{5}{6}]) = H(Y|X)$$

regardless of the value of $p$

Therefore, $I(X;Y) = H(Y) - H(Y|X) = 0$.