ECS452 2014/1 Part A.2 Dr.Prapun

B Frequency-Domain Analysis: Fourier Transform

Electrical engineers live in the two worlds, so to speak, of time and frequency. Frequency-domain analysis is an extremely valuable tool to the communications engineer, more so perhaps than to other systems analysts. Since the communications engineer is concerned primarily with signal bandwidths and signal locations in the frequency domain, rather than with transient analysis, the essentially steady-state approach of the (complex exponential) **Fourier series** and **transforms** is used rather than the Laplace transform.

B.1 Math background

B.1. ***Euler's formula***: $\boxed{e^{jx} = \cos x + j\sin x.}$

$$\cos(A) = \operatorname{Re}\left\{e^{jA}\right\} = \frac{1}{2}\left(e^{jA} + e^{-jA}\right)$$
$$\sin(A) = \operatorname{Im}\left\{e^{jA}\right\} = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right).$$

[16]Again, these are called the **law/rule of the lazy statistician** (LOTUS) [16, Thm 3.6 p 48],[7, p. 149] because it is so much easier to use the above formula than to first find the pmf of $g(X)$ or $g(X,Y)$. It is also called **substitution rule** [15, p 271].

B.2. We can use $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$ and $\sin x = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$ to derive many trigonometric identities:

(a) $\cos(-x) = \cos(x)$,

(b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$,

(c) $\cos^2(x) = \frac{1}{2}\left(\cos(2x) + 1\right)$

(d) $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and

(e) $\frac{d}{dx}\sin x = \cos x$

(f) **product-to-sum formula**

$$\cos(x)\cos(y) = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right). \tag{54}$$

B.2 Continuous-Time Fourier Transform

Definition B.3. The (direct) **Fourier transform** of a signal $g(t)$ is defined by

$$G(f) = \int_{-\infty}^{+\infty} g(t)e^{-j2\pi ft}dt \tag{55}$$

This provides the frequency-domain description of $g(t)$. Conversion back to the time domain is achieved via the **inverse (Fourier) transform**:

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft}df \tag{56}$$

- We may combine (55) and (56) into one compact formula:

$$\boxed{\int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft}df = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi ft}dt.} \tag{57}$$

- We may simply write $G = \mathcal{F}\{g\}$ and $g = \mathcal{F}^{-1}\{G\}$.

- Note that $G(0) = \int_{-\infty}^{\infty} g(t)dt$ and $g(0) = \int_{-\infty}^{\infty} G(f)df$.

B.4. In some references[17], the (direct) Fourier transform of a signal $g(t)$ is defined by

$$\hat{G}(\omega) = \int_{-\infty}^{+\infty} g(t)e^{-j\omega t}dt \tag{58}$$

In which case, we have

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{G}(\omega)\, e^{j\omega t}d\omega = g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \hat{G}(\omega) = \int\limits_{-\infty}^{\infty} g(t)\, e^{-j\omega t}dt \tag{59}$$

- In `MATLAB`, these calculations are carried out via the commands `fourier` and `ifourier`.
- Note that $\hat{G}(0) = \int\limits_{-\infty}^{\infty} g(t)dt$ and $g(0) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{G}(\omega)d\omega$.
- The relationship between $G(f)$ in (55) and $\hat{G}(\omega)$ in (58) is given by

$$G(f) = \hat{G}(\omega)\Big|_{\omega=2\pi f} \tag{60}$$

$$\hat{G}(\omega) = G(f)|_{f=\frac{\omega}{2\pi}} \tag{61}$$

Before we introduce our first but crucial transform pair in Example B.7 which will involve rectangular function, we want to introduce the indicator function which gives compact representation of the rectangular function. We will see later that the transform of the rectangular function gives a sinc function. Therefore, we will also discuss the sinc function as well.

Definition B.5. An **indicator function** gives only two values: 0 or 1. It is usually written in the form

$$1[\text{some condition(s) involving } t].$$

Its value at a particular t is one if and only if the condition(s) inside is satisfied for that t. For example,

$$1[|t| \le a] = \begin{cases} 1, & -a \le t \le a, \\ 0, & \text{otherwise.} \end{cases}$$

[17]MATLAB uses this definition.

Alternatively, we can use a set to specify the values of t at which the indicator function gives the value 1:

$$1_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases}$$

In particular, the set A can be some interval:

$$1_{[-a,a]}(t) = \begin{cases} 1, & -a \le t \le a, \\ 0, & \text{otherwise}. \end{cases}$$

Definition B.6. The function $\text{sinc}(x) \equiv (\sin x)/x$ is plotted in Figure 24.

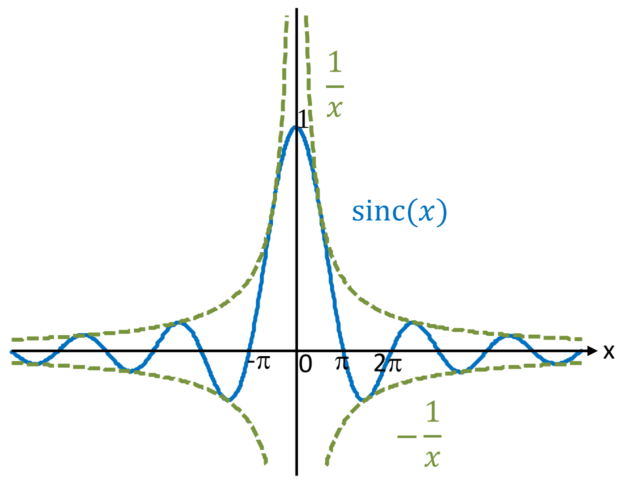

Figure 24: Sinc function

- This function plays an important role in signal processing. It is also known as the filtering or interpolating function.
- Using L'Hôpital's rule, we find $\lim_{x\to 0} \text{sinc}(x) = 1$.
- $\text{sinc}(x)$ is the product of an oscillating signal $\sin(x)$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, $\text{sinc}(x)$ exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.
- In `MATLAB` and in [17, eq. 2.64], $\text{sinc}(x)$ is defined as $(\sin(\pi x))/\pi x$. In which case, it is an even damped oscillatory function with zero crossings at integer values of its argument.

Example B.7. Rectangular function[18] and Sinc function:

$$1\left[|t| \leq a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{\sin(2\pi f a)}{\pi f} = \frac{2\sin(a\omega)}{\omega} = 2a\ \operatorname{sinc}(a\omega) \tag{62}$$

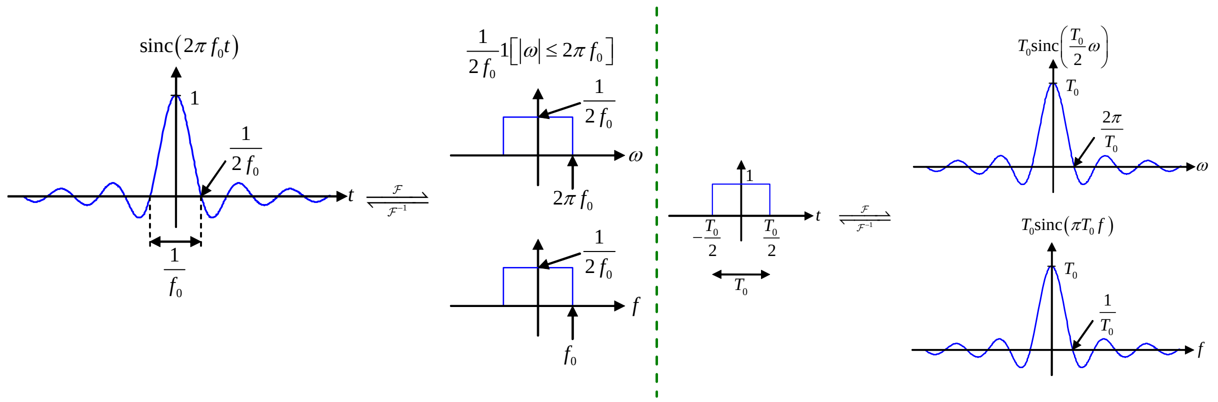

Figure 25: Fourier transform of sinc and rectangular functions

- By setting $a = T_0/2$, we have

$$1\left[|t| \leq \frac{T_0}{2}\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} T_0 \operatorname{sinc}(\pi T_0 f). \tag{63}$$

- In [9, p 78], the function $1\left[|t| \leq 0.5\right]$ is defined as the **unit gate** function $\operatorname{rect}(x)$.

Definition B.8. The (Dirac) **delta function** or (unit) impulse function is denoted by $\delta(t)$. It is usually depicted as a vertical arrow at the origin. Note that $\delta(t)$ is not a true function; it is undefined at $t = 0$. We define $\delta(t)$ as a generalized function which satisfies the **sampling property** (or **sifting property**)

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \tag{64}$$

for any function $\phi(t)$ which is continuous at $t = 0$.

- In this way, the delta "function" has no mathematical or physical meaning unless it appears under the operation of integration.

[18]

- Intuitively we may visualize $\delta(t)$ as an infinitely tall, infinitely narrow rectangular pulse of unit area: $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}1\left[|t|\le\frac{\varepsilon}{2}\right]$.

B.9. Properties of $\delta(t)$:

- $\delta(t) = 0$ for $t \neq 0$.
  $\delta(t-T) = 0$ for $t \neq T$.
- $\int_A \delta(t)dt = 1_A(0)$.
  (a) $\int \delta(t)dt = 1$.
  (b) $\int_{\{0\}} \delta(t)dt = 1$.
  (c) $\int_{-\infty}^{x} \delta(t)dt = 1_{[0,\infty)}(x)$. Hence, we may think of $\delta(t)$ as the "derivative" of the unit step function $U(t) = 1_{[0,\infty)}(t)$.
- $\int \phi(t)\delta(t-c)dt = \phi(c)$ for ϕ continuous at T. In fact, for any $\varepsilon > 0$,

$$\int_{T-\varepsilon}^{T+\varepsilon} \phi(t)\delta(t-c)dt = \phi(c).$$

- Convolution property:

$$(\delta * \phi)(t) = (\phi * \delta)(t) = \int_{-\infty}^{\infty} \phi(\tau)\delta(t-\tau)d\tau = \phi(t) \tag{65}$$

  where we assume that ϕ is continuous at t.
- $\delta(at) = \frac{1}{|a|}\delta(t)$. In particular,

$$\delta(\omega) = \frac{1}{2\pi}\delta(f) \tag{66}$$

  and

$$\delta(\omega-\omega_0) = \delta(2\pi f - 2\pi f_0) = \frac{1}{2\pi}\delta(f-f_0), \tag{67}$$

  where $\omega = 2\pi f$ and $\omega_0 = 2\pi f_0$.

Example B.10. Fourier transform pairs involving the δ function:

(a) $\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1$.

(b) $e^{j2\pi f_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \delta(f-f_0)$.

(c) $e^{j\omega_0 t} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi\delta\left(\omega-\omega_0\right)$.

(d) $\cos(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2}\left(\delta\left(f-f_0\right)+\delta\left(f+f_0\right)\right)$.

B.11. Symmetry Properties:

(a) **Conjugate symmetry**[19]: If $g(t)$ is **real**-valued, then $G(-f) = (G(f))^*$

- Observe that if we know $G(f)$ for all f positive, we also know $G(f)$ for all f negative. Interpretation: Only half of the spectrum contains all of the information. Positive-frequency part of the spectrum contains all the necessary information. The negative-frequency half of the spectrum can be determined by simply complex conjugating the positive-frequency half of the spectrum.

(b) If $g(t)$ is real and even, then so is $G(f)$.

(c) If $g(t)$ is real and odd, then $G(f)$ is pure imaginary and odd.

B.12. ***Shifting*** properties

(a) ***Time-shift***:

$$g\left(t-t_1\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} e^{-j2\pi f t_1} G\left(f\right)$$

- Note that $|e^{-j2\pi f t_1}| = 1$. So, the (amplitude) spectrum of $g\left(t-t_1\right)$ looks exactly the same as the spectrum of $g(t)$ (unless you also look at their phases).

(b) ***Frequency-shift*** (or modulation):

$$e^{j2\pi f_1 t} g\left(t\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G\left(f-f_1\right)$$

[19]This is called Hermitian symmetry in [13, p 17].

B.13. Let $g(t)$, $g_1(t)$, and $g_2(t)$ denote signals with $G(f)$, $G_1(f)$, and $G_2(f)$ denoting their respective Fourier transforms.

(a) ***Superposition theorem*** (linearity):

$$a_1 g_1(t) + a_2 g_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} a_1 G_1(f) + a_2 G_2(f).$$

(b) ***Scale-change*** theorem (scaling property [9, p 88]):

$$g(at) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{|a|} G\left(\frac{f}{a}\right).$$

- The function $g(at)$ represents the function $g(t)$ *compress*ed in time by a factor a (when $|a| > l$). Similarly, the function $G(f/a)$ represents the function $G(f)$ *expand*ed in frequency by the same factor a.
- The scaling property says that if we "squeeze" a function in t, its Fourier transform "stretches out" in f. It is not possible to arbitrarily concentrate both a function and its Fourier transform.
- Generally speaking, the more concentrated $g(t)$ is, the more spread out its Fourier transform $G(f)$ must be.
- This trade-off can be formalized in the form of an *uncertainty principle*. See also B.19 and B.20.
- Intuitively, we understand that compression in time by a factor a means that the signal is varying more rapidly by the same factor. To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a. Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed.

(c) ***Duality theorem*** (Symmetry Property [9, p 86]):

$$G(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} g(-f).$$

- In words, for any result or relationship between $g(t)$ and $G(f)$, there exists a dual result or relationship, obtained by interchanging the roles of $g(t)$ and $G(f)$ in the original result (along with some minor modifications arising because of a sign change).

  In particular, if the Fourier transform of $g(t)$ is $G(f)$, then the Fourier transform of $G(f)$ with f replaced by t is the original time-domain signal with t replaced by $-f$.
- If we use the ω-definition (58), we get a similar relationship with an extra factor of 2π:

$$G_2(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2\pi g(-\omega).$$

Example B.14. From Example B.7, we know that

$$1\left[|t| \le a\right] \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 2a \ \operatorname{sinc}\left(2\pi a f\right) \tag{68}$$

By the duality theorem, we have

$$2a \operatorname{sinc}(2\pi a t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1[|-f| \le a],$$

which is the same as

$$\operatorname{sinc}(2\pi f_0 t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} \frac{1}{2f_0} 1[|f| \le f_0]. \tag{69}$$

Both transform pairs are illustrated in Figure 25.

Definition B.15. The **convolution** of two signals, $g_1(t)$ and $g_2(t)$, is a new function of time, $g(t)$. We write

$$g = g_1 * g_2.$$

It is defined as the integral of the product of the two functions after one is reversed and shifted:

$$g(t) = (g_1 * g_2)(t) \tag{70}$$
$$= \int_{-\infty}^{+\infty} g_1(\mu) g_2(t-\mu) d\mu = \int_{-\infty}^{+\infty} g_1(t-\mu) g_2(\mu) d\mu. \tag{71}$$

- Note that t is a parameter as far as the integration is concerned.

- The integrand is formed from g_1 and g_2 by three operations:
  (a) time reversal to obtain $g_2(-\mu)$,
  (b) time shifting to obtain $g_2(-(\mu - t)) = g_2(t - \mu)$, and
  (c) multiplication of $g_1(\mu)$ and $g_2(t - \mu)$ to form the integrand.
- In some references, (70) is expressed as $g(t) = g_1(t) * g_2(t)$.

B.16. Convolution theorem:

(a) Convolution-in-time rule:

$$g_1 * g_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_1 \times G_2. \tag{72}$$

(b) Convolution-in-frequency rule:

$$g_1 \times g_2 \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_1 * G_2. \tag{73}$$

Example B.17. We can use the convolution theorem to "prove" the frequency-sift property in B.12.

B.18. Parseval's theorem (Rayleigh's energy theorem, Plancherel formula) for Fourier transform:

$$\int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df. \tag{74}$$

The LHS of (74) is called the (total) **energy** of $g(t)$. On the RHS, $|G(f)|^2$ is called the **energy spectral density** of $g(t)$. By integrating the energy spectral density over all frequency, we obtain the signal 's total energy. The energy contained in the frequency band B can be found from the integral $\int_B |G(f)|^2 df$.

More generally, Fourier transform preserves the inner product [3, Theorem 2.12]:

$$\langle g_1, g_2 \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt = \int_{-\infty}^{\infty} G_1(f) G_2^*(f) df = \langle G_1, G_2 \rangle.$$

B.19. (Heisenberg) **Uncertainty Principle** [3, 14]: Suppose g is a function which satisfies the normalizing condition $\|g\|_2^2 = \int |g(t)|^2 dt = 1$ which automatically implies that $\|G\|_2^2 = \int |G(f)|^2 df = 1$. Then

$$\left(\int t^2 |g(t)|^2 dt\right)\left(\int f^2 |G(f)|^2 df\right) \geq \frac{1}{16\pi^2}, \tag{75}$$

and equality holds if and only if $g(t) = Ae^{-Bt^2}$ where $B > 0$ and $|A|^2 = \sqrt{2B/\pi}$.

- In fact, we have

  $$\left(\int t^2 |g(t-t_0)|^2 dt\right)\left(\int f^2 |G(f-f_0)|^2 df\right) \geq \frac{1}{16\pi^2},$$

  for every t_0, f_0.

- The proof relies on Cauchy-Schwarz inequality.

- For any function h, define its dispersion Δ_h as $\frac{\int t^2 |h(t)|^2 dt}{\int |h(t)|^2 dt}$. Then, we can apply (75) to the function $g(t) = h(t)/\|h\|_2$ and get

  $$\Delta_h \times \Delta_H \geq \frac{1}{16\pi^2}.$$

B.20. A signal cannot be simultaneously time-limited and band-limited.