ECS452 2014/1 Part I.3 Dr.Prapun

2.4 (Shannon) Entropy for Discrete Random Variables

General idea

Entropy is a measure of uncertainty of a random variable [3, p 13].

Entropy quantifies/measures the amount of uncertainty (randomness) in a RV.

It arises as the answer to a number of natural questions. One such question that will be important for us is "What is the average length of the shortest *description* of the random variable?"

Definition 2.41. The **entropy** $H(X)$ of a discrete random variable X is defined by

Recall: $\log_2 a = \frac{\ln a}{\ln 2}$

$\log_2 1 = 0$

$$H(X) = -\sum_{x \in \mathcal{S}_X} p_X(x) \log_2 p_X(x) = -\mathbb{E}\left[\log_2 p_X(X)\right].$$

$0 \underbrace{\log_2 0}_{-\infty}$

$\mathbb{E}[g(X)] = \sum_x p_X(x) g(x)$ In this case, $g(x) = -\log_2 p_X(x)$

- The log is to the base 2 and entropy is expressed in bits (per symbol).
  - The base of the logarithm used in defining H can be chosen to be any convenient real number $b > 1$ but if $b \neq 2$ the unit will not be in bits.
  - If the base of the logarithm is e (10), the entropy is measured in nats (hartley [Hart]).
  - Unless otherwise specified, base 2 is our default base.
- Based on continuity arguments, we shall assume that $0 \ln 0 = 0$.

$\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} \overset{\text{L'Hôspital's}}{=} \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} (-x) = 0$

$= \frac{1}{4}\log_2 2^{-2} = -\frac{2}{4}\underbrace{\log_2 2}_{1} = -\frac{1}{2}$

Back then, the probability values are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}$

Example 2.42. The entropy of the random variable X in Example 2.31 is 1.75 bits (per symbol).

$-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{8}\log_2\frac{1}{8} = \frac{1}{2}+\frac{1}{2}+\frac{3}{8}+\frac{3}{8}$

$= 1.75$ bits (/symbol)

Example 2.43. The entropy of a fair coin toss is 1 bit (per toss).

The probabilities involved are $\frac{1}{2}, \frac{1}{2}$

$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = \frac{1}{2}+\frac{1}{2} = 1$ bit (/symbol)

2.44. Note that entropy is a functional of the (unordered probability values in the) pmf of X. It does not depend on the actual values taken by the random variable X, but only on the (unordered) probabilities. Therefore, sometimes, we write $H(p_X)$ instead of $H(X)$ to emphasize this fact. Moreover, because we use only the probability values, we can use the row vector representation $\underline{p}$ of the pmf p_X and simply express the entropy as $H(\underline{p})$.

In MATLAB, to calculate $H(X)$, we may define a row vector `pX` from the pmf p_X. Then, the value of the entropy is given by

$\vec{a} = \begin{pmatrix} \vdots \end{pmatrix}$

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HX = -pX*(log2(pX))'.
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$\rightarrow \underline{a} = (\cdot\ \cdot\ \cdot\ \cdot)$

Example 2.45. The entropy of a uniform (discrete) random variable X on $\{1, 2, 3, \ldots, n\}$:

$$p_X(x) = \begin{cases} 1/n, & x = 1, 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

$$H(X) = -\sum_x p_X(x)\log_2 p_X(x) = -n \times \frac{1}{n}\log_2\frac{1}{n} = \log_2 n$$

Alternatively,

$$H(X) = -\mathbb{E}\left[\log_2 p_X(X)\right] = -\mathbb{E}\left[\log_2\frac{1}{n}\right] = -\log_2\frac{1}{n} = \log_2 n$$

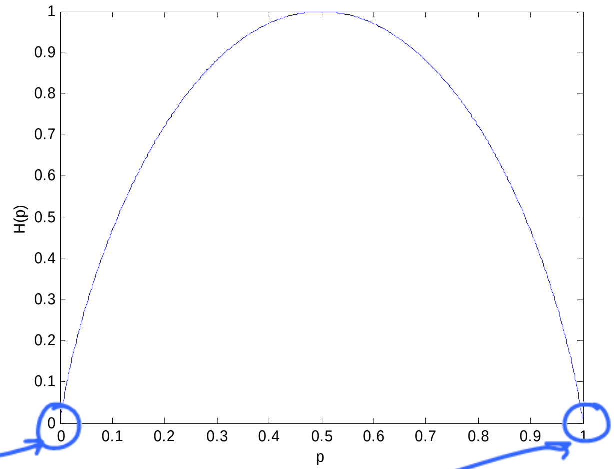
Example 2.46. The entropy of a Bernoulli random variable X:

$$p_X(x) = \begin{cases} p, & x = 1, \\ 1-p, & x = 0, \\ 0, & \text{otherwise} \end{cases} \qquad H(X) = -p\log_2 p - (1-p)\log_2(1-p)$$

Binary RV

$$p_X(x) = \begin{cases} p, & x = a, \\ 1-p, & x = b, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.47. **Binary Entropy Function** : We define $h_b(p)$, $h(p)$ or $H(p)$ to be $-p\log p - (1-p)\log(1-p)$, whose plot is shown in Figure 3.



When $p = 0$ or 1, X becomes degenerated/deterministic

$\Rightarrow$ no randomness

$\Rightarrow H(X) = 0$.

Figure 3: Binary Entropy Function

2.48. Two important facts about entropy:

(a) $H(X) \le \log_2 |S_X|$ with equality if and only if X is a uniform random variable.

(b) $H(X) \ge 0$ with equality if and only if X is not random.

In summary,

$$\underset{\text{deterministic}}{0} \overset{(b)}{\le} H(X) \overset{(a)}{\le} \underset{\text{uniform}}{\log_2 |S_X|}.$$

Theorem 2.49. The expected length $\mathbb{E}[\ell(X)]$ of any uniquely decodable binary code for a random variable X is greater than or equal to the entropy $H(X)$; that is,

$$\mathbb{E}[\ell(X)] \ge H(X)$$

with equality if and only if $2^{-\ell(x)} = p_X(x)$. [3, Thm. 5.3.1]

Definition 2.50. Let $L(c, X)$ be the expected codeword length when random variable X is encoded by code c.

Let $L^*(X)$ be the minimum possible expected codeword length when random variable X is encoded by a uniquely decodable code c:

$$L^*(X) = \min_{\text{UD } c} L(c, X).$$

2.51. Given a random variable X, let c_{Huffman} be the Huffman code for this X. Then, from the optimality of Huffman code mentioned in 2.37,

$$L^*(X) = L(c_{\text{Huffman}}, X).$$

Theorem 2.52. The optimal code for a random variable X has an expected length less than $H(X) + 1$:

$$L^*(X) < H(X) + 1.$$

2.53. Combining Theorem 2.49 and Theorem 2.52, we have true for Huffman code.

true for any UD code

$$H(X) \leq L^*(X) < H(X) + 1. \tag{3}$$

Definition 2.54. Let $L_n^*(X)$ be the minimum expected codeword length per symbol when the random variable X is encoded with n-th extension uniquely decodable coding. Of course, this can be achieve by using n-th extension Huffman coding.

2.55. An extension of (3):

$$H(X) \leq L_n^*(X) < H(X) + \frac{1}{n}. \tag{4}$$

In particular,

$$\lim_{n\to\infty} L_n^*(X) = H(X).$$

In otherwords, by using large block length, we can achieve an expected length per source symbol that is arbitrarily close to the value of the entropy.

2.56. Operational meaning of entropy: Entropy of a random variable is the average length of its shortest description.

2.57. References

- Section 16.1 in Carlson and Crilly [2]
- Chapters 2 and 5 in Cover and Thomas [3]
- Chapter 4 in Fine [4]
- Chapter 14 in Johnson, Sethares, and Klein [6]
- Section 11.2 in Ziemer and Tranter [16]