Quiz 1

Thursday, June 27, 2013 1:52 PM

Quiz 1. Consider six vectors ... $\vec{v}^{(1)} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$ $\vec{v}^{(2)} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ $\vec{v}^{(3)} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

$\vec{v}^{(4)} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ $\vec{v}^{(5)} = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$ $\vec{v}^{(6)} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

Use GSOP to find $\vec{u}^{(1)}, \vec{u}^{(2)}, \dots$

(The $\vec{v}^{(i)}$ should be use in the same order as given above when they are put in GSOP.)

solution

$$\vec{u}^{(1)} = \vec{v}^{(1)} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}. \qquad \|\vec{u}^{(1)}\|^2 = 2$$

$$\vec{u}^{(2)} = \vec{v}^{(2)} - \underbrace{\mathrm{proj}_{\vec{u}^{(1)}} \vec{v}^{(2)}}_{} = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \begin{pmatrix}1/2\\1/2\\0\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\\1\\0\end{pmatrix}. \qquad \|\vec{u}^{(2)}\|^2 = \frac{1}{4}+\frac{1}{4}+1 = \frac{1}{2}+1 = \frac{3}{2}$$

$$\frac{\langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle}{\|\vec{u}^{(1)}\|^2} \vec{u}^{(1)} = \frac{\left\langle \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \right\rangle}{2} = \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\0\\0\end{pmatrix}$$

$$\vec{u}^{(3)} = \vec{v}^{(3)} - \underbrace{\mathrm{proj}_{\vec{u}^{(1)}} \vec{v}^{(3)}}_{} - \underbrace{\mathrm{proj}_{\vec{u}^{(2)}} \vec{v}^{(3)}}_{} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1\\-1\\2\\0\end{pmatrix} = \frac{1}{6}\left(\begin{pmatrix}6\\0\\0\\6\end{pmatrix} - \begin{pmatrix}3\\3\\0\\0\end{pmatrix} - \begin{pmatrix}1\\-1\\2\\0\end{pmatrix}\right)$$

$$= \frac{1}{6}\begin{pmatrix}2\\-2\\-2\\6\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}$$

$$= \frac{\langle \vec{v}^{(3)}, \vec{u}^{(1)} \rangle}{\|\vec{u}^{(1)}\|^2} \vec{u}^{(1)} = \frac{\left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \right\rangle}{2} \vec{u}^{(1)} = \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$= \mathrm{proj}_{\underbrace{2\vec{u}^{(2)}}} \vec{v}^{(3)} = \frac{\left\langle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\\0\end{pmatrix} \right\rangle}{6} 2\vec{u}^{(2)} = \frac{1}{6}\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}$$

$$2\vec{u}^{(2)} = \begin{pmatrix}1\\-1\\2\\0\end{pmatrix} \qquad \|2\vec{u}^{(2)}\|^2 = 1+1+4 = 6$$

$$\left[\mathrm{proj}_{k\vec{u}} \vec{v} = \frac{\langle \vec{v}, k\vec{u} \rangle}{\|k\vec{u}\|^2} = \frac{k^*}{|k|^2} \frac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{u}\|^2} k\vec{u} = \mathrm{proj}_{\vec{u}} \vec{v}\right]$$

$$\vec{u}^{(4)} = \underbrace{\vec{v}^{(4)}}_{=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}} - \underbrace{\mathrm{proj}_{\underbrace{\vec{u}^{(1)}}_{=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}} \vec{v}^{(4)}}_{\frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} - \underbrace{\mathrm{proj}_{\underbrace{2\vec{u}^{(2)}}_{=\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}}} \vec{v}^{(4)}}_{\frac{1}{6}\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}} - \underbrace{\mathrm{proj}_{\underbrace{3\vec{u}^{(3)}}_{=\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}}} \vec{v}^{(4)}}_{-\frac{2}{12}\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}}$$

. . . . .

$$\frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} \qquad \frac{1}{6}\begin{pmatrix}1\\-1\\2\\0\end{pmatrix} \qquad -\frac{2}{12}\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}$$

$$= \frac{1}{12}\left(\begin{pmatrix}0\\12\\12\\0\end{pmatrix} - \begin{pmatrix}6\\6\\0\\0\end{pmatrix} - \begin{pmatrix}2\\-2\\4\\0\end{pmatrix} + \begin{pmatrix}2\\-2\\-2\\6\end{pmatrix}\right) = \frac{1}{12}\begin{pmatrix}-6\\6\\6\\6\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}$$

Normally, we would proceed with $\vec{v}^{(5)}$ and $\vec{v}^{(6)}$. However, at this point, we have four orthogonal vectors:

$$\vec{u}^{(1)} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \vec{u}^{(2)} = \frac{1}{2}\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}, \quad \vec{u}^{(3)} = \frac{1}{3}\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}, \text{ and } \vec{u}^{(4)} = \frac{1}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}.$$

Note that the $\vec{v}^{(i)}$ have four components and hence the maximum size of their orthonormal basis is 4. Therefore, there is no need to look further. We know that we will get $\vec{0}$ when we work with $\vec{v}^{(5)}$ and $\vec{v}^{(6)}$.

In conclusion, from GSOP, we get four orthogonal vectors

$$\vec{u}^{(1)} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \vec{u}^{(2)} = \frac{1}{2}\begin{pmatrix}1\\-1\\2\\0\end{pmatrix}, \quad \vec{u}^{(3)} = \frac{1}{3}\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}, \text{ and } \vec{u}^{(4)} = \frac{1}{2}\begin{pmatrix}-1\\1\\1\\1\end{pmatrix}.$$