4 Projection and GSOP

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12:41 PM

Projection and GSOP

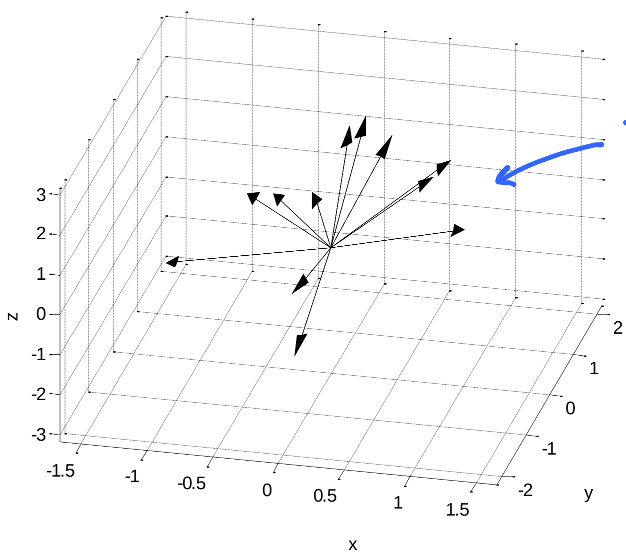
↳ start with many vectors:

$\vec{v}^{(1)}, \vec{v}^{(2)}, \vec{v}^{(3)}, \ldots, \vec{v}^{(M)}$

If each of them is 3D,
then you need $3 \times M$ values to talk
about these vectors.

GSOP → similar to "compression"
Reducing dimension.

Main idea:
"The M vectors that we had originally, they may be linearly dependent."



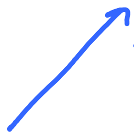
This may look like the vectors are in 3D.
However, they are all on the same plane.
So, if we look at them from the correct perspective, they can be referenced to using only two coordinates.

Back to GSOP,

Step 1: (creation of $\vec{u}^{(1)}$)

$\vec{v}^{(1)}$

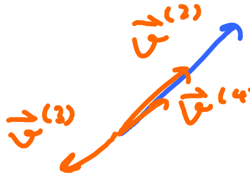
$\vec{u}^{(1)} = \vec{v}^{(1)}$



$\vec{v}^{(1)}$

$\vec{u}^{(1)} = \vec{v}^{(1)}$

[Remark:

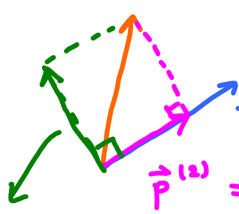


$\vec{v}^{(2)}$ $\vec{v}^{(1)}$ $\vec{v}^{(4)}$ $\vec{v}^{(3)}$

$\leftarrow$ For this example all $\vec{v}^{(i)}$'s are simply $c\,\vec{u}^{(1)}$]

Step 2: (creation of $\vec{u}^{(2)}$)

Projection:



$\vec{v}^{(2)}$

$\vec{u}^{(1)} = \vec{v}^{(1)}$

$$\vec{p}^{(2)} = \text{proj}_{\vec{u}^{(1)}} \vec{v}^{(2)} = \frac{\langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle}{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle} \vec{u}^{(1)} = c\,\vec{u}^{(1)}$$

$$\vec{u}^{(2)} = \vec{o}^{(2)} = \vec{v}^{(2)} - \vec{p}^{(2)}$$
$$= \vec{v}^{(2)} - \text{proj}_{\vec{u}^{(1)}} \vec{v}^{(2)}$$

Some interesting proof:

Want to show that $\vec{o}^{(2)} \perp \vec{p}^{(2)}$

will show $\vec{o}^{(2)} \perp \vec{u}^{(1)}$

$$\langle \vec{o}^{(2)}, \vec{u}^{(1)} \rangle = \langle \vec{v}^{(2)} - c\,\vec{u}^{(1)}, \vec{u}^{(1)} \rangle = \langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle - c \langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle$$

$$\left[(\vec{a} \pm \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} \pm \vec{b} \cdot \vec{c} \right]$$

$$= \langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle - \frac{\langle \vec{v}^{(2)}, \vec{u}^{(1)} \rangle}{\cancel{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle}} \cancel{\langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle}$$

$$= 0$$

$$\langle \vec{u}^{(1)}, \vec{o}^{(2)} \rangle = \langle \vec{u}^{(1)}, \vec{v}^{(2)} - c\,\vec{u}^{(1)} \rangle = \langle \vec{u}^{(1)}, \vec{v}^{(2)} \rangle - c^{*} \langle \vec{u}^{(1)}, \vec{u}^{(1)} \rangle$$

$$= \langle \vec{w}^{(1)}, \vec{v}^{(3)} \rangle - \langle \vec{v}^{(3)}, \vec{w}^{(1)} \rangle^* = 0.$$

The proof above demonstrates one important point:
when we work with complex-valued vector, factoring a constant out of the innerproduct needs some extra care.

Real-valued vectors:

$$(c_1 \vec{a} + c_2 \vec{b}) \cdot \vec{c} = c_1 (\vec{a} \cdot \vec{c}) + c_2 (\vec{b} \cdot \vec{c})$$

$$(\vec{a}, c_1 \vec{b} + c_2 \vec{c}) = c_1 (\vec{a} \cdot \vec{b}) + c_2 (\vec{a} \cdot \vec{c})$$

complex-valued vectors:

$$\langle c_1 \vec{a} + c_2 \vec{b}, \vec{c} \rangle = c_1 \langle \vec{a}, \vec{c} \rangle + c_2 \langle \vec{b}, \vec{c} \rangle$$

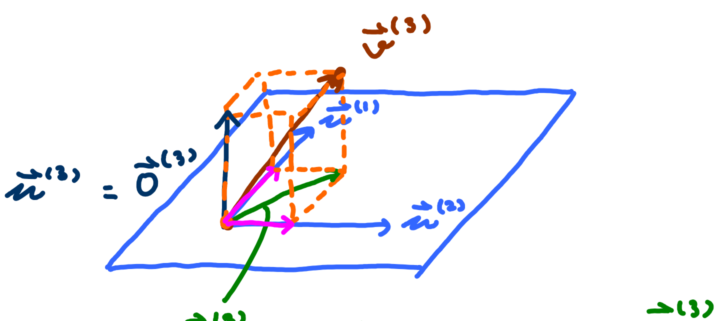
$$\langle \vec{a}, c_1 \vec{b} + c_2 \vec{c} \rangle = c_1^* \langle \vec{a}, \vec{b} \rangle + c_2^* \langle \vec{a}, \vec{c} \rangle$$

The reason for the extra conjugation operation:

$$\langle \vec{a}, c\vec{b} \rangle \neq c \langle \vec{a}, \vec{b} \rangle \quad \text{(in general)}$$

$$\langle \vec{a}, c\vec{b} \rangle = (c\vec{b})^H \vec{a} = c^* \vec{b}^H \vec{a} = c^* \langle \vec{a}, \vec{b} \rangle$$

step 3: (creation of $\vec{w}^{(3)}$)



$$\vec{p}^{(3)} = \text{proj}_{\text{span}\{\vec{w}^{(1)}, \vec{w}^{(2)}\}} \vec{v}^{(3)}$$

$$= \text{proj}_{\vec{w}^{(1)}} \vec{v}^{(3)} + \text{proj}_{\vec{w}^{(2)}} \vec{v}^{(3)}$$

$$\vec{w}^{(3)} = \vec{o}^{(3)} = \vec{v}^{(3)} - \vec{p}^{(3)} = \vec{v}^{(3)} - (\; \ldots \;)$$

If we have more than 3 vectors $(M > 3)$, then we simply extend the above procedure.

What do we get out of GSOP?

(1) $\{\vec{v}^{(1)}, \vec{v}^{(2)}, \ldots, \vec{v}^{(M)}\} \xrightarrow{\text{GSOP}} \{\vec{w}^{(1)}, \vec{w}^{(2)}, \ldots, \vec{w}^{(K)}\}$

Turn M vectors into K vectors.

$(K \leq M)$.

(2) Any $\vec{v}^{(i)}$ can be expressed as

$$\vec{v}^{(i)} = \sum_{j=1}^{i} c_j^{(i)} \vec{w}^{(j)} = \sum_{j=1}^{K} c_j^{(i)} \vec{w}^{(j)}$$

$\leq i$ if $K < M$

Let's assume $\vec{v}^{(i)}$ is N-D.

Usually, we use N numbers to talk about it.

Now, you only need K numbers $c_1^{(i)}, c_2^{(i)}, \ldots, c_K^{(i)}$

summarized into one vector

$$\vec{c}^{(i)} = \begin{pmatrix} c_1^{(i)} \\ c_2^{(i)} \\ \vdots \\ c_K^{(i)} \end{pmatrix}$$

$$\vec{v}^{(i)} = \sum_{j=1}^{K} c_j^{(i)} \vec{w}^{(j)}$$

If we want $c_k^{(i)}$

$$\langle \vec{v}^{(i)}, \vec{w}^{(k)} \rangle = \sum_{j=1}^{K} c_j^{(i)} \langle \vec{w}^{(j)}, \vec{w}^{(k)} \rangle$$

$$= c_k^{(i)} \langle \vec{w}^{(k)}, \vec{w}^{(k)} \rangle$$

$$c_k^{(i)} = \frac{\langle \vec{v}^{(i)}, \vec{w}^{(k)} \rangle}{\langle \vec{w}^{(k)}, \vec{w}^{(k)} \rangle}$$