Introduction to OFDM Systems

Dr. Prapun Suksompong

prapun@siit.tu.ac.th

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Outline

1. Overview of OFDM technique
2. Wireless Channel
3. Multi-carrier Transmission
4. Implementation: DFT and FFT
5. Concluding remarks
6. More advanced topics (if ∃ interest and time permitted)
   i. Cyclic Prefix (CP) and Circular Convolution
   ii. OFDM Drawbacks: PAPR and its solutions
   iii. OFDM-based Multiple Access Systems
Video Attendance Check

- Say your name into the camera 😊
- Make sure that your voice is loud enough
OFDM: Overview

- Let \( S_1, S_2, \ldots, S_N \) be the information symbol.
- The discrete baseband OFDM modulated symbol can be expressed as

\[
s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi kt}{T_s} \right), \quad 0 \leq t \leq T_s
\]

Note that:

\[
\text{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left( \text{Re}\{S_k\} \cos\left( \frac{2\pi kt}{T_s} \right) - \text{Im}\{S_k\} \sin\left( \frac{2\pi kt}{T_s} \right) \right)
\]
OFDM Application

- 802.11 Wi-Fi: a and g versions
- DVB-T (the terrestrial digital TV broadcast system used in most of the world outside North America)
- DMT (the standard form of ADSL - Asymmetric Digital Subscriber Line)
- WiMAX

<table>
<thead>
<tr>
<th>Wireless</th>
<th>Wireline</th>
</tr>
</thead>
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<tr>
<td>IEEE 802.11a, g, n (WiFi) Wireless LANs</td>
<td>ADSL and VDSL broadband access via POTS copper wiring</td>
</tr>
<tr>
<td>IEEE 802.15.3a Ultra Wideband (UWB) Wireless PAN</td>
<td>MoCA (Multi-media over Coax Alliance) home networking</td>
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<tr>
<td>IEEE 802.16d, e (WiMAX), WiBro, and HiperMAN Wireless MANs</td>
<td>PLC (Power Line Communication)</td>
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<tr>
<td>IEEE 802.20 Mobile Broadband Wireless Access (MBWA)</td>
<td></td>
</tr>
<tr>
<td>DVB (Digital Video Broadcast) terrestrial TV systems: DVB-T, DVB-H, T-DMB, and ISDB-T</td>
<td></td>
</tr>
<tr>
<td>DAB (Digital Audio Broadcast) systems: EUREKA 147, Digital Radio Mondiale, HD Radio, T-DMB, and ISDB-TSB</td>
<td></td>
</tr>
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<td>Flash-OFDM cellular systems</td>
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</tr>
<tr>
<td>3GPP UMTS &amp; 3GPP@ LTE (Long-Term Evolution) and 4G</td>
<td></td>
</tr>
</tbody>
</table>
Single-User OFDM

In this talk, we shall focus on the single user case of OFDM.
Motivation

Why do we need OFDM?

- First, we study the wireless channel.
- There are a couple of difficult problems in communication system over wireless channel.
- Also want to achieve high data rate (throughput)
1. Wireless Channel
Single Carrier Transmission

- **Baseband:**
  \[
  s(t) = \sum_{k=0}^{N-1} S_k p(t - kT_s)
  \]
  
  \[
  p(t) = 1_{[0,T_s]}(t) = \begin{cases} 
  1, & t \in [0,T_s) \\
  0, & \text{otherwise}.
  \end{cases}
  \]

- **Passband:**
  \[
  x(t) = \text{Re}\left\{ s(t) e^{j2\pi f_c t} \right\} = s(t) \cos(2\pi f_c t)
  \]

Rectangular waveform

Carrier frequency

Valid when \( s(t) \) is real-valued
Frequency-Domain Analysis

Shifting Properties:

\[ g(t - t_0) \xrightarrow{\mathcal{F}} e^{-j2\pi f t_0} G(f) \]

\[ e^{j2\pi f t_0} g(t) \xrightarrow{\mathcal{F}} G(f - f_0) \]

Modulation:

\[ m(t)\cos(2\pi f_c t) \xrightarrow{\mathcal{F}} \frac{1}{2} M(f - f_c) + \frac{1}{2} M(f + f_c) \]
Can you sketch $|P(f)|$?

$$p(t) = A \times 1_{[t \in [0,T]}}$$

$$[S_k] = [-1,-1,1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,-1,1,1,-1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,-1,1,1,
This is also the spectrum of \( c(t - kT) \) for any \( k \).

\[
s(t) = \sum_{k=0}^{N-1} S_k p(t - kT) \xrightarrow{\mathcal{F}} S(f) = P(f) \sum_{k=0}^{n-1} S_k e^{-j2\pi fkT}
\]

\( p(t) = A \times 1[t \in [0,T)] \)

\[
[S_k] = [-1,-1,1,-1,1,1,-1,-1,1,-1,-1,1,-1,1,-1,-1,1,-1,1,-1,1,-1,1,1]
\]
Multipath Propagation

- In a wireless mobile communication system, a transmitted signal propagating through the wireless channel often encounters multiple reflective paths until it reaches the receiver.
- We refer to this phenomenon as **multipath propagation** and it causes fluctuation of the amplitude and phase of the received signal.
- We call this fluctuation **multipath fading**.
Wireless Comm. and Multipath Fading

The signal received consists of a number of reflected rays, each characterized by a different amount of attenuation and delay.

\[ r(t) = x(t) * h(t) + n(t) = \sum_{i=0}^{v} \beta_i x(t - \tau_i) + n(t) \]

\[ h(t) = \sum_{i=0}^{v} \beta_i \delta(t - \tau_i) \]

\[ h_1(t) = 0.5\delta(t) + 0.2\delta(t - 0.2T_s) + 0.3\delta(t - 0.3T_s) + 0.1\delta(t - 0.5T_s) \quad \text{Weak} \]

\[ h_2(t) = 0.5\delta(t) + 0.2\delta(t - 0.7T_s) + 0.3\delta(t - 1.5T_s) + 0.1\delta(t - 2.3T_s) \quad \text{Strong} \]
Frequency Domain

The **transmitted** signal (envelope)

Channel with **weak** multipath

Channel with **strong** multipath
Tradeoff

- We want weak multipath fading.
- So, we want to make the symbol period (T) large with respect to the delay(s).
- But larger symbol period means lower rate!
- Hence, there is a tradeoff between the raw data rate and the quality of the received signal.
- Alternatively, we may allow the system to experience strong multipath fading.
  - Use complicated equalizer at the receiver.
  - Not our approach.
COST 207 Channel Model

- Based on channel measurements with a bandwidth of 8–10 MHz in the 900 MHz band used for 2G systems such as GSM.

<table>
<thead>
<tr>
<th>Path #</th>
<th>Rural Area (RA)</th>
<th>Typical Urban (TU)</th>
<th>Bad Urban (BU)</th>
<th>Hilly Terrain (HT)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Delay (µs)</td>
<td>Power (dB)</td>
<td>Delay (µs)</td>
<td>Power (dB)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>-4</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>-8</td>
<td>0.5</td>
<td>-2</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>-12</td>
<td>1.6</td>
<td>-6</td>
</tr>
<tr>
<td>5</td>
<td>0.4</td>
<td>-16</td>
<td>2.3</td>
<td>-8</td>
</tr>
<tr>
<td>6</td>
<td>0.5</td>
<td>-20</td>
<td>5.0</td>
<td>-10</td>
</tr>
</tbody>
</table>

[Fazel and Kaiser, 2008, Table 1-1]
### 3GPP LTE Channel Modelss

<table>
<thead>
<tr>
<th>Path number</th>
<th>Extended Pedestrian A (EPA)</th>
<th>Extended Vehicular A (EVA)</th>
<th>Extended Typical Urban (ETU)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Delay (ns)</td>
<td>Power (dB)</td>
<td>Delay (ns)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>-1</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>70</td>
<td>-2</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>90</td>
<td>-3</td>
<td>310</td>
</tr>
<tr>
<td>5</td>
<td>110</td>
<td>-8</td>
<td>370</td>
</tr>
<tr>
<td>6</td>
<td>190</td>
<td>-17.2</td>
<td>710</td>
</tr>
<tr>
<td>7</td>
<td>410</td>
<td>-20.8</td>
<td>1090</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>1730</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td>2510</td>
</tr>
</tbody>
</table>

[Fazel and Kaiser, 2008, Table 1-3]
3GPP 6-tap typical urban (TU6)

- Delay profile and frequency response of 3GPP 6-tap typical urban (TU6) Rayleigh fading channel in 5 MHz band.

[3GPP TS 45.005 – 3GPP; Technical Specification Group GSM/EDGE Radio Access Network; Radio Transmission and Reception (Release 7)]
Equalization

- Chapter 11 of [Goldsmith, 2005]
- Delay spread causes ISI
- In a broad sense, equalization defines any signal processing technique used at the receiver to alleviate the ISI problem caused by delay spread. [Goldsmith, 2005]
- Higher data rate applications are more sensitive to delay spread, and generally require high-performance equalizers or other ISI mitigation techniques.
- Signal processing can also be used at the transmitter to make the signal less susceptible to delay spread.
  - Ex. spread spectrum and multicarrier modulation
Equalizer design

- Balance ISI mitigation with noise enhancement
  - Both the signal and the noise pass through the equalizer
- Nonlinear equalizers suffer less from noise enhancement than linear equalizers, but typically entail higher complexity.
- Most equalizers are implemented digitally after A/D conversion
  - Such filters are small, cheap, easily tuneable, and very power efficient.
- The optimal equalization technique is maximum likelihood sequence estimation (MLSE).
  - Unfortunately, the complexity of this technique grows exponentially with the length of the delay spread, and is therefore impractical on most channels of interest.
- Viterbi algorithm
Simple Analog Equalizer

- Remove all ISI
- Disadvantages:
  - If some frequencies in the channel frequency response $H(f)$ are greatly attenuated, the equalizer $H_{eq}(f) = 1 / H(f)$ will greatly enhance the noise power at those frequencies.
  - If the channel frequency response $H(f)$ has a spectral null ($= 0$ for some frequency), then the power of the new noise is infinite.
  - Even though the ISI effects are (completely) removed, the equalized system will perform poorly due to its greatly reduced SNR.
Wireless Propagation

[Bahai, 2002, Fig. 2.1]
Three steps towards modern OFDM

1. Solve Multipath Problem
   \( \rightarrow \) Multicarrier modulation (FDM)

2. Gain Spectral Efficiency
   \( \rightarrow \) Orthogonality of the carriers

3. Achieve Efficient Implementation
   \( \rightarrow \) FFT and IFFT
OFDM

2. Multi-Carrier Transmission
Single-Carrier Transmission

Input Data

Digital Modulator

I

Q

Pulse-Shaping Filter

X

FS

f_1

Σ

90°

X

Power Amplifier

Up Converter

FS - Frequency Synthesizer
f_1 - the first carrier frequency
I - In-phase component
Q - Quadrature component

[Karim and Sarraf, 2002, Fig 3-1]
Multi-Carrier Transmission

- Convert a serial high rate data stream on to **multiple parallel low rate** sub-streams.
- Each sub-stream is modulated on its own sub-carrier.
- **Time domain perspective**: Since the symbol rate on each sub-carrier is much less than the initial serial data symbol rate, the effects of delay spread, i.e. ISI, significantly decrease, reducing the complexity of the equalizer.

\[ T_s = N_c T_d \]

[Fazel and Kaiser, 2008, Fig 1-4]
Frequency Division Multiplexing

- **Frequency Domain Perspective**: Even though the fast fading is frequency-selective across the entire OFDM signal band, it is effectively flat in the band of each low-speed signal.

[Myung and Goodman, 2008] [The flatness assumption is the same one that you used in Riemann approximation of integral.]
Frequency Division Multiplexing

- To facilitate separation of the signals at the receiver, the carrier frequencies were *spaced sufficiently far apart* so that the signal spectra did not overlap. Empty spectral regions between the signals assured that they could be separated with readily realizable filters.
- The resulting spectral efficiency was therefore quite low.

![Frequency Division Multiplexing Diagram](image)
## Multi-Carrier (FDM) vs. Single Carrier

<table>
<thead>
<tr>
<th>Single Carrier</th>
<th>Multi-Carrier (FDM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single higher rate serial scheme</td>
<td>Parallel scheme. Each of the parallel subchannels can carry a low signalling rate, proportional to its bandwidth.</td>
</tr>
</tbody>
</table>

- **Multipath problem:** Far more susceptible to inter-symbol interference (ISI) due to the short duration of its signal elements and the higher distortion produced by its wider frequency band
- **Complicated equalization**

- **Long duration signal elements and narrow bandwidth in sub-channels.**
- **Complexity problem:** If built straightforwardly as several \((N)\) transmitters and receivers, will be more costly to implement.
- **BW efficiency problem:** The sum of parallel signalling rates is less than can be carried by a single serial channel of that combined bandwidth because of the unused guard space between the parallel sub-carriers.
FDM (con’t)

- Before the development of equalization, the parallel technique was the preferred means of achieving high rates over a dispersive channel, in spite of its high cost and relative bandwidth inefficiency.
OFDM

- OFDM = Orthogonal frequency division multiplexing
- One of multi-carrier modulation (MCM) techniques
  - Parallel data transmission (of many sequential streams)
  - A broadband is divided into many narrow sub-channels
  - Frequency division multiplexing (FDM)
- High spectral efficiency
  - The sub-channels are made orthogonal to each other over the OFDM symbol duration $T_s$.
    - Spacing is carefully selected.
  - Allow the sub-channels to overlap in the frequency domain.
  - Allow sub-carriers to be spaced as close as theoretically possible.
Orthogonality

• Two vectors/functions are **orthogonal** if their **inner product** is zero.

• The symbol \( \perp \) is used to denote orthogonality.

**Vector:**
\[
\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}^* = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{k=1}^{n} a_k b_k = 0
\]

**Time-domain:**
\[
\langle a, b \rangle = \int_{-\infty}^{\infty} a(t) b^*(t) dt = 0
\]

**Frequency domain:**
\[
\langle A, B \rangle = \int_{-\infty}^{\infty} A(f) B^*(f) df = 0
\]

**Example:**
\[
2t + 3 \text{ and } 5t^2 + t - \frac{17}{9} \text{ on } [-1, 1]
\]

**Example:**
\[
\sin \left( 2\pi k_1 \frac{t}{T} \right) \text{ and } \cos \left( 2\pi k_2 \frac{t}{T} \right) \text{ on } [0, T]
\]
\[
e^{j2\pi n \frac{t}{T}} \text{ on } [0, T]
\]
Orthogonality in Communication

CDMA

\[ s(t) = \sum_{k=0}^{\ell-1} S_k c_k(t) \xrightarrow{\mathcal{F}} S(f) = \sum_{k=0}^{\ell-1} S_k C_k(f) \]

where \( c_{k_1} \perp c_{k_2} \)

TDMA

\[ s(t) = \sum_{k=0}^{\ell-1} S_k c(t - kT_s) \xrightarrow{\mathcal{F}} S(f) = C(f) \sum_{k=0}^{\ell-1} S_k e^{-j2\pi fkT_s} \]

where \( c(t) \) is time-limited to \([0,T]\).

This is a special case of CDMA with \( c_k(t) = c(t - kT_s) \)

The \( c_k \) are non-overlapping in time domain.

FDMA

\[ S(f) = \sum_{k=0}^{\ell-1} S_k C(f - k\Delta f) \]

where \( C(f) \) is frequency-limited to \([0,\Delta f]\).

This is a special case of CDMA with \( C_k(f) = C(f - k\Delta f) \)

The \( C_k \) are non-overlapping in freq. domain.
OFDM

- Let $S_1, S_2, \ldots, S_N$ be the information symbol.
- The discrete baseband OFDM modulated symbol can be expressed as

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp \left( j \frac{2\pi kt}{T_s} \right), \quad 0 \leq t \leq T_s$$

$$= \sum_{k=0}^{N-1} S_k \frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \exp \left( j \frac{2\pi kt}{T_s} \right)$$

Another special case of CDMA!

Note that:

$$\text{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left( \text{Re}\{S_k\} \cos \left( \frac{2\pi kt}{T_s} \right) - \text{Im}\{S_k\} \sin \left( \frac{2\pi kt}{T_s} \right) \right)$$
OFDM: Orthogonality

\[
\int_{0}^{T_s} c_{k_1}(t) c_{k_2}^*(t) dt = \int_{0}^{T_s} \exp \left( j \frac{2\pi k_1 t}{T_s} \right) \exp \left( -j \frac{2\pi k_2 t}{T_s} \right) dt
\]

= \int_{0}^{T_s} \exp \left( j \frac{2\pi (k_1 - k_2) t}{T_s} \right) dt = \begin{cases} T_s, & k_1 = k_2 \\ 0, & k_1 \neq k_2 \end{cases}

When \( k_1 = k_2 \),

\[
\int_{0}^{T_s} c_{k_1}(t) c_{k_2}^*(t) dt = \int_{0}^{T_s} 1 dt = T_s
\]

When \( k_1 \neq k_2 \),

\[
\int_{0}^{T_s} c_{k_1}(t) c_{k_2}^*(t) dt = \left. \frac{T_s}{j2\pi(k_1 - k_2)} \exp \left( j \frac{2\pi (k_1 - k_2) t}{T_s} \right) \right|_{0}^{T_s}
\]

= \frac{T_s}{j2\pi(k_1 - k_2)} (1 - 1) = 0
Frequency Spectrum

\[
 s(t) = \sum_{k=0}^{N-1} S_k c_k(t) \quad \rightarrow \quad S(f) = \sum_{k=0}^{N-1} S_k C_k(f)
\]

\[
 c(t) = \frac{1}{\sqrt{N}} 1_{[0,T_s]}(t) \quad \rightarrow \quad C(f) = \frac{1}{\sqrt{N}} T_s e^{-j2\pi f \frac{T_s}{2}} \sin c(\pi T_s f)
\]

\[
 c_k(t) = c(t) \exp\left( j \frac{2\pi kt}{T_s} \right) \quad \rightarrow \quad C_k(f) = C\left( f - \frac{k}{T_s} \right) = C\left( f - k\Delta f \right)
\]

\[
 \Delta f = \frac{1}{T_s}
\]

This is the term that makes the technique FDM.
Subcarrier Spacing

Each QAM signal carries one of the original input complex numbers.

The spectrum of each QAM signal is of the form with nulls at the center of the other subcarriers.

\[ s(t) = \sum_{k=0}^{N-1} S_k \sqrt{\frac{1}{N}} I_{[0,T_s]}(t) \exp\left( j\frac{2\pi kt}{T_s} \right) \]

\[ S(f) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k e^{-j2\pi(f-k\Delta f)\frac{T_s}{T}} \sin c\left( \pi T_s (f - k\Delta f) \right) \]

\[ \Delta f = \frac{1}{T_s} \]

N separate QAM signals, at N frequencies separated by the signalling rate.
Normalized Power Density Spectrum

Flatter when have more sub-carriers

[Fazel and Kaiser, 2008, Fig 1-5]
Time-Domain Signal

Real component of an OFDM signal

Imaginary component of an OFDM signal

[Bahai, 2002, Fig 1.7]

Real and Imaginary components of an OFDM symbol is the superposition of several harmonics modulated by data symbols.

\[ s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp \left( j \frac{2\pi kt}{T_s} \right), \quad 0 \leq t \leq T_s \]

\[ \text{Re}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left( \text{Re}\{S_k\} \cos \left( \frac{2\pi kt}{T_s} \right) \right) \]

\[ \text{Im}\{s(t)\} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left( \text{Im}\{S_k\} \sin \left( \frac{2\pi kt}{T_s} \right) \right) \]

in-phase part

quadrature part
Summary

- So, we have a scheme which achieve
  - Large symbol duration ($T_s$) and hence less multipath problem
  - Good spectral efficiency

- One more problem:
  - There are so many carriers!
3. Implementation: DFT and FFT
Discrete Fourier Transform (DFT)

Transmitter produces

\[ s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi kt}{T_s} \right), \quad 0 \leq t \leq T_s \]

Sample the signal in time domain every \( T_s / N \) gives

\[ s[n] = s\left(n \frac{T_s}{N}\right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi k}{\frac{T_s}{N}} n \right) \]

\[ = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left( j \frac{2\pi kn}{N} \right) = \sqrt{N} \text{IDFT}\{S\}[n] \]

We can implement OFDM in the discrete domain!
Discrete Fourier Transform (DFT)

In DFT, we work with $N$-point signal (finite-length sequence of length $N$) in both time and frequency domain. To simplify the definition we define

$$\psi_N = e^{\frac{j2\pi}{N}}$$

and the DFT matrix $Q = \Psi_N$ whose element on the $p$th row and $q$th column is given by $\psi_N^{-(p-1)(q-1)}$.

Key Property:

$$\Psi_N^{-1} = \frac{1}{N} \Psi_N^*.$$ Equivalently, $\Psi_N^{-1} \Psi_N = NI_N$.

$$\frac{1}{\sqrt{N}} \Psi_N$$ is a unitary matrix.
**DFT**

**Definition 5.3.** The $N$-point DFT of an $N$-point signal (column vector) $x$ is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jnk\frac{2\pi}{N}} = \sum_{n=0}^{N-1} x[n] \psi^{-nk}_N ; 0 \leq k < N.$$ 

The inverse DFT is given by

$$x[n] \overset{DFT-1}{\leftrightarrow} \frac{1}{N} \sum_{k=0}^{N-1} X[k] \psi^{nk}_N \quad \overset{\text{DFT}}{\leftrightarrow} \quad X[k] = \sum_{n=0}^{N-1} x[n] \psi^{-nk}_N$$

In matrix form,

$$x = \frac{1}{N} \Psi^*_N X \quad \overset{DFT}{\leftrightarrow} \quad X = \Psi_N \times x.$$
**DFT**

**Definition 5.3.** The \(N\)-point DFT of an \(N\)-point signal (column vector) \(x\) is given by

\[
X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} x[n] \psi_{N}^{-nk} ; \quad 0 \leq k < N.
\]

The inverse DFT is given by

\[
x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \psi_{N}^{nk} \quad \xrightarrow{\text{DFT}} \quad X[k] = \sum_{n=0}^{N-1} x[n] \psi_{N}^{-nk}
\]

In matrix form,

\[
x = \frac{1}{N} \Psi_{N}^{*} X \quad \xrightarrow{\text{DFT}} \quad X = \Psi_{N} \times x.
\]
DFT: Example

Digitized Basis Functions for a 16 point DFT

16 samples of real part of basis function for 16 pt. DFT

Pick one of the 16 basis functions

$e^{\alpha n/16}$

$e^{\alpha 2n/16}$

$e^{\alpha 3n/16}$

$e^{\alpha 4n/16}$

$e^{\alpha 5n/16}$

$e^{\alpha 6n/16}$

$e^{\alpha 7n/16}$

$e^{\alpha 8n/16}$

Real part of Basis Function

16 samples of imag. part of basis function for 16 pt. DFT

$-j^{\alpha n/16}$

$-j^{\alpha 2n/16}$

$-j^{\alpha 3n/16}$

$-j^{\alpha 4n/16}$

$-j^{\alpha 5n/16}$

$-j^{\alpha 6n/16}$

$-j^{\alpha 7n/16}$

$-j^{\alpha 8n/16}$

Imaginary part of Basis Function

An $N$-point FFT requires only on the order of $N \log N$ multiplications, rather than $N^2$ as in a straightforward computation.
FFT

- The history of the FFT is complicated.
- As with many discoveries and inventions, it arrived before the (computer) world was ready for it.
- Usually done with $N$ a power of two.
  - Not only is it very efficient in terms of computing time, but is ideally suited to the binary arithmetic of digital computers.
  - From the implementation point of view it is better to have, for example, a FFT size of 1024 even if only 600 outputs are used than try to have another length for FFT between 600 and 1024.

DFT Samples

Here are the points $s[n]$ on the continuous-time version $s(t)$:

$$s(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi k t}{T_s}\right), \quad 0 \leq t \leq T_s$$

$$s[n] = s\left(n \frac{T_s}{N}\right)$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kn}{N}\right)$$

$$= \sqrt{N} \ \text{IDFT}\{S\}[n]$$

$$0 \leq n < N$$
Oversampling
Oversampling (2)

- Increase the number of sample points from \( N \) to \( LN \) on the interval \([0, T_s]\).

- \( L \) is called the over-sampling factor.

\[
s[n] = s\left(n \frac{T_s}{N}\right), \quad 0 \leq n < N
\]

\[
s^{(L)}[n] = s\left(n \frac{T_s}{LN}\right), \quad 0 \leq n < LN
\]

\[
s^{(L)}[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kn}{LN}\right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kn}{LN}\right)
\]

\[
= \frac{1}{\sqrt{N}} LN \left( \frac{1}{LN} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kn}{LN}\right) \right)
\]

\[
= L\sqrt{N} \left( \frac{1}{LN} \sum_{k=0}^{N-1} S_k \exp\left(j \frac{2\pi kn}{LN}\right) + \sum_{k=N}^{NL-1} 0 \exp\left(j \frac{2\pi kn}{LN}\right) \right)
\]

\[
= L\sqrt{N} \left( \frac{1}{LN} \sum_{k=0}^{NL-1} \tilde{S}_k \exp\left(j \frac{2\pi kn}{LN}\right) \right) = L\sqrt{N} \text{IDFT}\{\tilde{S}\}[n]
\]

Zero padding:

\[
\tilde{S}_k = \begin{cases} S_k, & 0 \leq k < N \\ 0, & N \leq k < LN \end{cases}
\]
Oversampling: Summary

\[ s[n] = s \left( n \frac{T_s}{N} \right) = \sqrt{N} \text{IDFT}\{S\}[n] \quad 0 \leq n < N \]

\[ s^{(L)}[n] = s \left( n \frac{T_s}{LN} \right) = L\sqrt{N} \text{IDFT}\{\tilde{S}\}[n] \quad 0 \leq n < LN \]

Zero padding:
\[ \tilde{S}_k = \begin{cases} S_k, & 0 \leq k < N \\ 0, & N \leq k < LN \end{cases} \]
OFDM implementation by IFFT/FFT

\[ s^{(L)}[n] = s\left(n \frac{T_s}{LN}\right) = L\sqrt{N} \text{IFFT}^{(L)}\{\tilde{S}\}[n] \]

This form of OFDM is often referred to as \textbf{Discrete Multi-Tone (DMT)}. 

\[ r[n] = \sqrt{N} \text{IFFT}\{S\}[n] \]

\[ R_k = \sqrt{N} S_k \]
OFDM with Memoryless Channel

\[ h(t) = \beta \delta(t) \]

[should be \( h(t) = \beta \delta(t - \tau) \)]

\[ y(t) = h(t) * s(t) + w(t) = \beta s(t) + w(t) \]

Additive white Gaussian noise

Sample every \( T_s/N \)

\[ y[n] = \beta s[n] + w[n] \]

FFT

\[ s[n] = \sqrt{N} \text{IFFT}\{S\}[n] \]

\[ Y_k = \text{FFT}\{y\}[n] = \beta \sqrt{N} S_k + W_k \]

Sub-channel are independent.

(No ICI)
Channel with Finite Memory

Discrete time baseband model:

\[ y[n] = \{ h * s \}[n] + w[n] = \sum_{m=0}^{\nu} h[m] s[n-m] + w[n] \]

where \( h[n] = 0 \) for \( n < 0 \) and \( n > \nu \)

\[ w[n] \sim \mathcal{C}\mathcal{N}(0, N_0) \]

We will assume that \( \nu \ll N \)

Remarks:

\( Z = X + jY \) is a complex Gaussian if \( X \) and \( Y \) are jointly Gaussian.

If \( X, Y \) is i.i.d. \( \mathcal{N}(0, \sigma^2) \), then \( Z = X + iY \sim \mathcal{C}\mathcal{N}(0, \sigma_Z^2) \) where \( \sigma_Z^2 = 2\sigma^2 \) with

\[ f_z(z) = f_{X,Y}(\text{Re}\{z\}, \text{Im}\{z\}) = \frac{1}{\pi \sigma_Z^2} e^{-\frac{|z|^2}{\sigma_Z^2}}. \]
OFDM Architecture

[Bahai, 2002, Fig 1.11]
4. Remarks about OFDM
Summary: OFDM Advantages

- For a given channel delay spread, the implementation complexity is much lower than that of a conventional single carrier system with time domain equalizer.

- **Spectral efficiency** is high since it uses overlapping orthogonal subcarriers in the frequency domain.

- Modulation and demodulation are implemented using inverse discrete Fourier transform (IDFT) and discrete Fourier transform (DFT), respectively, and fast Fourier transform (FFT) algorithms can be applied to make the overall system efficient.

- **Capacity** can be significantly increased by adapting the data rate per subcarrier according to the signal-to-noise ratio (SNR) of the individual subcarrier.
Example: 802.11a

<table>
<thead>
<tr>
<th>Parameter</th>
<th>IEEE 802.11a</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth</td>
<td>20 MHz</td>
</tr>
<tr>
<td>Number of sub-carriers $N_c$</td>
<td>52 (48 data + 4 pilots) (64 FFT)</td>
</tr>
<tr>
<td>Symbol duration $T_s$</td>
<td>4 $\mu$s</td>
</tr>
<tr>
<td>Carrier spacing $F_s$</td>
<td>$312.5$ kHz [= \frac{1}{4-0.8[\mu s]} ]</td>
</tr>
<tr>
<td>Guard time $T_g$</td>
<td>0.8 $\mu$s</td>
</tr>
<tr>
<td>Modulation</td>
<td>BPSK, QPSK, 16-QAM, and 64-QAM</td>
</tr>
<tr>
<td>FEC coding</td>
<td>Convolutional with code rate 1/2 up to 3/4</td>
</tr>
<tr>
<td>Max. data rate</td>
<td>54 Mbit/s</td>
</tr>
</tbody>
</table>
OFDM Drawbacks

- **High peak-to-average power ratio (PAPR)**
  - The transmitted signal is a superposition of all the subcarriers with different carrier frequencies and high amplitude peaks occur because of the superposition.

- **High sensitivity to frequency offset:**
  - When there are frequency offsets in the subcarriers, the orthogonality among the subcarriers breaks and it causes intercarrier interference (ICI).

- **A need for an adaptive or coded scheme to overcome spectral nulls in the channel**
  - In the presence of a null in the channel, there is no way to recover the data of the subcarriers that are affected by the null unless we use rate adaptation or a coding scheme.
OFDM

5. Cyclic Prefix (CP)
Three steps towards modern OFDM

1. Mitigate Multipath (ISI) → Multicarrier modulation (FDM)
2. Gain Spectral Efficiency → Orthogonality of the carriers
3. Achieve Efficient Implementation → FFT and IFFT
   • Completely eliminate ISI and ICI → Cyclic prefix
Cyclic Prefix: Motivation (1)

- Recall: Multipath Fading and Delay Spread
Cyclic Prefix: Motivation (2)

- When the number of sub-carriers increases, the OFDM symbol duration $T_s$ becomes large compared to the duration of the impulse response $\tau_{\text{max}}$ of the channel, and the amount of ISI reduces.
- Can we “eliminate” the multipath (ISI) problem?
- To reduce the ISI, add guard interval larger than that of the estimated delay spread.
- If the guard interval is left empty, the orthogonality of the sub-carriers no longer holds, i.e., ICI (inter-channel interference) still exists.
- To prevent both the ISI as well as the ICI, OFDM symbol is cyclically extended into the guard interval.

![Diagram showing cyclic prefix]

[Jiang]
Cyclic Prefix

Guard Interval, $T_{cp} > \tau_{max}$
Using empty spaces as guard interval at the beginning of each symbol

End of symbol is prepended to beginning
Guard interval still equals to $T_{cp}$

Using cyclic prefix:
OFDM symbol length: $T_{sym} + T_{cp}$
Efficiency: $T_{sym} / (T_{sym} + T_{cp})$
Convolution

1. Flip
2. Shift
3. Multiply
4. Add

\[(x \ast h)[n] = \sum_{m} x[m] h[n - m]\]
Circular Convolution

Replicate $x$ (now it looks periodic)
Then, perform the usual convolution only on $n = 0$ to $N-1$
Circular Convolution: Example

Find

\[ [1 \ 2 \ 3] \ast [4 \ 5 \ 6] \]

\[ [1 \ 2 \ 3] \circledast [4 \ 5 \ 6] \]

\[ [1 \ 2 \ 3 \ 0 \ 0] \circledast [4 \ 5 \ 6 \ 0 \ 0] \]
Discussion

- Circular convolution can be used to find the regular convolution by zero-padding.
- In modern OFDM, it is another way around.
- CTFT: convolution in time domain corresponds to multiplication in frequency domain.
- DFT: circular convolution in (discrete) time domain corresponds to multiplication in (discrete) frequency domain.
  - We want to have multiplication in frequency domain.
  - So, we want circular convolution and not the regular convolution.
- Real channel does regular convolution.
- With cyclic prefix, regular convolution can be used to create circular convolution.
Example

- Suppose $x^{(1)} = [1 -2 3 1 2]$ and $h = [3 2 1]$
- $[1 -2 3 1 2] \odot [3 2 1 0 0] = [8 -2 6 7 11]$
- $[1 2 1 -2 3 1 2] \times [3 2 1] = [3 8 8 -2 6 7 11 5 2]$

- Suppose $x^{(2)} = [2 1 -3 -2 1]$
- $[2 1 -3 -2 1] \odot [3 2 1 0 0] = [6 8 -5 -11 -4]$
- $[-2 1 2 1 -3 -2 1] \times [3 2 1] = [-6 -1 6 8 -5 -11 -4 0 1]$
- $[1 2 1 -2 3 1 2 -2 1 2 1 -3 -2 1] \times [3 2 1] = [3 8 8 -2 6 7 11 5 2] + [-6 -1 6 8 -5 -11 -4 0 1]$
- $= [3 8 8 -2 6 7 11 -1 1 6 8 -5 -11 -4 0 1]$
Circular Convolution: Key Properties

• Consider an $N$-point signal $x[n]$

• **Cyclic Prefix (CP) insertion**: If $x[n]$ is extended by copying the last $\nu$ samples of the symbols at the beginning of the symbol:

$$
\tilde{x}[n] = \begin{cases} 
x[n], & 0 \leq n \leq N - 1 \\
x[n + N], & -\nu \leq n \leq -1 
\end{cases}
$$

• Key Property 1:

$$
\{h \ast x\}[n] = (h \ast \tilde{x})[n] \text{ for } 0 \leq n \leq N - 1
$$

• Key Property 2:

$$
\{h \ast x\}[n] \xrightarrow{\text{FFT}} H_k X_k
$$
OFDM with CP for Channel w/ Memory

- We want to send $N$ samples $S_0, S_1, \ldots, S_{N-1}$ across noisy channel with memory.
- First apply IFFT: $S_k \xrightarrow{\text{IFFT}} s[n]$
- Then, add cyclic prefix
  $$\hat{s} = [s[N-\nu], \ldots, s[N-1], s[0], \ldots, s[N-1]]$$
- This is inputted to the channel.
- The output is
  $$y[n] = [p[N-\nu], \ldots, p[N-1], r[0], \ldots, r[N-1]]$$
- Remove cyclic prefix to get $r[n] = h[n] \ast s[n] + w[n]$
- Then apply FFT: $r[n] \xrightarrow{\text{FFT}} R_k$
- By circular convolution property of DFT, $R_k = H_k S_k + W_k$
OFDM System Design: CP

- A good ratio between the CP interval and symbol duration should be found, so that all multipaths are resolved and not significant amount of energy is lost due to CP.
- As a thumb rule, the CP interval must be two to four times larger than the root mean square (RMS) delay spread.

[Tarokh, 2009, Fig 2.9]
Reference

OFDM

6. OFDM-Based Multiple Access
OFDM-based Multiple Access

- Three multiple access techniques
  1. OFDMA,
  2. OFDM-TDMA, and
  3. OFDM-CDMA
OFDM-TDMA

- Users are separated via **time slots**.
- A particular user is given **all** the subcarriers of the system for any specific OFDM symbol duration.
- All symbols allocated to all users are combined to form a OFDM-TDMA **frame**.

- Allows MS to reduce its power consumption, as the MS shall process only OFDM symbols which are dedicated to it.

- Since the OFDM-TDMA concept allocates the whole bandwidth to a single user, a reaction to different subcarrier attenuations could consist of leaving out highly distorted subcarriers.
OFDMA

- Available subcarriers are distributed among all the users for transmission at any time instant.
- The subcarrier assignment is made at least for a time frame.
- Based on the subchannel condition, different baseband modulation schemes can be used for the individual subchannels.
- The fact that each user experiences a different radio channel can be exploited by allocating only “good” subcarriers with high SNR to each user.
- The number of subchannels for a specific user can be varied, according to the required data rate.
OFDM-TDMA vs. OFDMA

- User 1
- User 2
- User 3

Subcarriers vs. Time

OFDM-TDMA

OFDMA
OFDMA Block Diagram

- Subcarrier and bit allocation
  - User 1 data
  - User 2 data
  - User K data

- Combined subcarrier, bit, and power allocation

- Channel state information for all K users
  - x N subcarriers/user

- Adaptive mod. 1
- Adaptive mod. 2
- Adaptive mod. N

- IFFT

- Add guard interval

- Freq. selective fading channel for User k

- Extract bits for User k
  - User k data

- Adaptive demod. 1
- Adaptive demod. 2
- Adaptive demod. N

- FFT

- Remove guard interval
OFDM-CDMA

- User data are spread over several subcarriers and/or OFDM symbols using spreading codes, and combined with signal from other users.

- Several users transmit over the same subcarrier. In essence this implies **frequency-domain spreading**, rather than time-domain spreading, as it is conceived in a DS-CDMA system.