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ECS 332: Principles of Communications **2019/1**

HW 4 — Due: September 27, 4 PM **Solution**

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 6 pages.

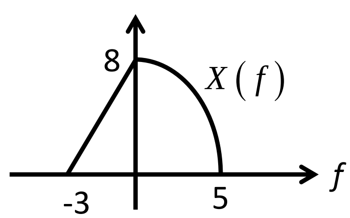
(b) (1 pt) Hard-copies are distributed in class. Original pdf file can be downloaded from the course website. Work and write your answers **directly on the provided hardcopy/file** (not on other blank sheet(s) of paper).

(c) (1 pt) Write your first name and the last three digits of your student ID in the spaces provided on the upper-right corner of this page. Furthermore, for online submission, your file name should start with your 10-digit student ID, followed by a space, the course code, a space, and the assignment number: "5565242231 332 HW4.pdf"

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1 (M2011)**.** The Fourier transform $X(f)$ for a signal $x(t)$ is shown in Figure 4.1.



Note that x(t) is not provided directly. Only its Fourier transform X(f) is plotted.

Figure 4.1: Plot of $X(f)$ for Problem 1.

Let $g(t) = x(-2t)$ and $y(t) = x(4-2t)$. Carefully sketch $|G(f)|$ and $|Y(f)|$.

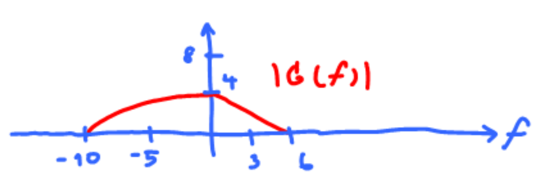
$= x(-2(t-2))$

$= x(at)$ where $a = -2$.

By the time-scaling property of Fourtier transform,

$G(f) = \frac{1}{|a|} X\left(\frac{f}{a}\right) = \frac{1}{2} X\left(-\frac{f}{2}\right)$

$a = -2$



Next, recall that time-shifting does not change the magnitude of the Fourier transform.
Hence, |Y(f)| = |G(f)|.

Hence,

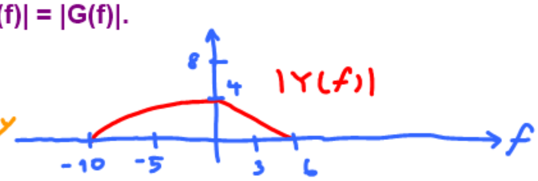
$|G(f)| = \left|\frac{1}{2} X\left(-\frac{f}{2}\right)\right| = \frac{1}{2}\left|X\left(-\frac{f}{2}\right)\right|.$

scaled vertically by a factor of $\frac{1}{2}$

no effect because $X(f)$ is already nonnegative.

time reversal

expanded horizontally by a factor of 2



aka. RF pulse, time-limited sinusoid, finite duration sinusoid. [C&C p. 59-60]

The main purpose of this problem is to see the spectrum of cosine pulse.

Problem 2. [1]

(a) Consider the cosine pulse

$2\pi(5)t \Rightarrow$ freq. = 5 $\Rightarrow$ From $t=-1$ to $t=1$, we should have $(1-(-1))\times 5 = 10$ cycles

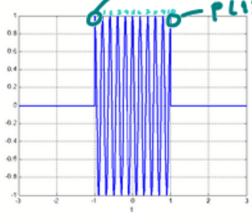
$$p(t) = \begin{cases} \cos(10\pi t), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Sketch $p(t)$ for $-3 \le t \le 3$.

$p(-1) = \cos(10\pi(-1)) = \cos(-10\pi) = 1$

$p(1) = \cos(10\pi(1)) = \cos(10\pi) = 1$

Here is a plot from MATLAB. Of course, the instruction says to sketch, so, your answer should be hand-written. However, it should be similar to this.



(ii) Find $P(f)$ analytically.

In general, consider the pulse of the form $p(t) = \begin{cases} \cos(2\pi f_0 t), & t_1 \le t \le t_2 \\ 0, & \text{otherwise.} \end{cases} = \cos(2\pi f_0 t) \times r(t)$

$r(t)$ is the rectangular pulse on the time interval $[t_1, t_2]$.

Writing it in this form makes it clear that we may view p(t) as a modulated signal where r(t) is the message.

Now we apply what we know about modulation:

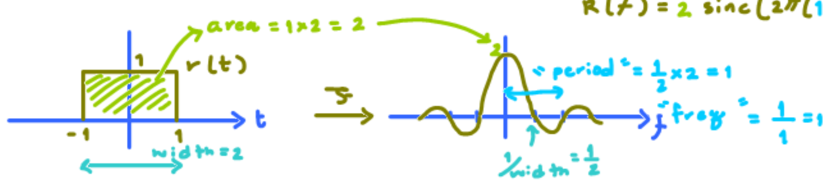
In time domain, $r(t)$ is multiplied by $\cos(2\pi f_0 t)$.

In freq. domain, $R(f)$ is shifted to $\pm f_0$ (and scaled by $\frac{1}{2}$).

In particular, $P(f) = \frac{1}{2}\left(R(f-f_0) + R(f+f_0)\right)$. ★

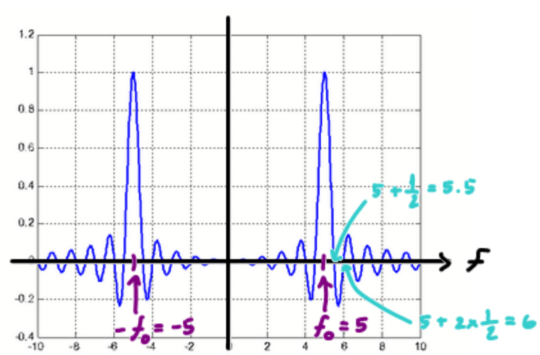
Here, $f_0 = 5$, $t_1 = -1$, and $t_2 = 1$.

★ $R(f) = 2\,\text{sinc}(2\pi(1)f) \Rightarrow P(f) = \text{sinc}(2\pi(f-5)) + \text{sinc}(2\pi(f+5))$



(iii) Sketch $P(f)$ from -10 Hz to 10 Hz.

Here is a plot from MATLAB. Of course, the instruction says to sketch, so, your answer should be hand-written. However, it should be similar to this.



[1] Inspired by [Carlson and Crilly, 2009, Q2.2-1 and Q2.2-2].

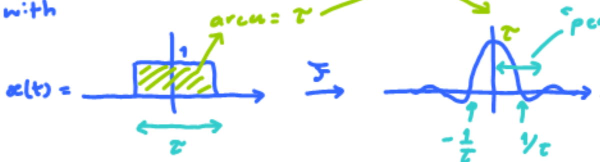
(b) Consider the cosine pulse

$$p(t) = \begin{cases} \cos(10\pi t), & 2 \le t \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Again, as in part a.ii, the main task here is to find R(f). Let's try to derive a general formula...

(i) Find $P(f)$ analytically.

start with



"period" of sinc $= 2 \times \frac{1}{\tau}$

$\Rightarrow$ freq. of sinc $= \frac{\tau}{2}$

$= X(f) = \tau \operatorname{sinc}\left(2\pi \frac{\tau}{2} f\right)$

Here, $\tau = t_2 - t_1$.

Observe that $r(t)$ is the time-shifted version of the $x(t)$ above: $r(t) = x\left(t - \frac{t_1+t_2}{2}\right)$

By the time-shift property,

$$R(f) = \underbrace{e^{-j\cancel{2}\pi f \frac{t_1+t_2}{\cancel{2}}}} X(f) = e^{-j\pi f(t_1+t_2)} (t_2-t_1) \operatorname{sinc}(\pi f (t_2-t_1)) \quad \star\star$$

Because $|\ |=1$, we know that $|R(f)| = |X(f)|$.

Here, $f_0 = 5$ (same), $t_1 = 2$, and $t_2 = 4$. Therefore, $R(f) = e^{-j6\pi f}\, 2 \operatorname{sinc}(2\pi f)$ $\star\star$ $\star$

and $P(f) = e^{-j6\pi(f-5)} \operatorname{sinc}(2\pi(f-5)) + e^{-j6\pi(f+5)} \operatorname{sinc}(2\pi(f+5))$ $\star$

(ii) Use `MATLAB`. Mimic the code in `specrect.m` to plot the spectrum of $p(t)$. Follow the settings below:

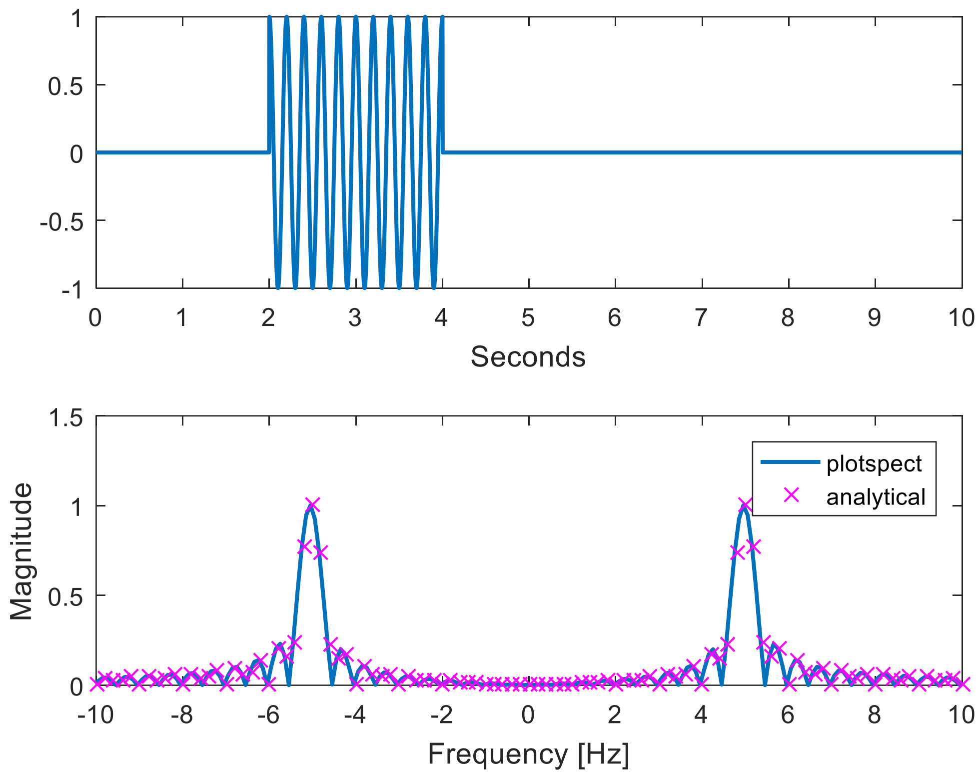
- Consider the time t from 0 to 10 [s] when you set up the time vector.
- Use the sampling frequency of 500 samples per sec. So, the sampling interval (the time between adjacent samples) is $T_s = 1/500$.
- With the above sampling frequency, `plotspect` will plot the magnitude spectrum from -250 to 250 Hz. Use the function `xlim` (or the magnifier glass GUI) to limit your frequency view to be only from -10 to +10 Hz.

(iii) Also in `MATLAB`, add the plot of your analytical answer from part (i) into the same figure as part (ii).

- Print this figure and attach it at the end of your HW.
- On this attached page, compare the two plots. (Write some description/observation. Are they the same? How can you tell?)

Caution: The built-in `sinc` function in `MATLAB` is defined using the normalized version. So, you will need to remove a factor of π from the argument of each sinc function found in part (i) when you type it into `MATLAB`.

Q3.b.ii The magnitude spectrum plot from the modified `specrect.m` is provided in the bottom part of the figure below.



Q3.b.iii In addition, the analytical expression in part (i) is plotted using the "x" marks on top of the provided plot from `specrect.m`.

The two plots generally agrees. However, small difference can be observed. The plot from plotspec seems to be shifted to the left by a small amount from the analytical prediction.

Problem 3. You are given the baseband signals (i) $m(t) = \cos 1000\pi t$; (ii) $m(t) = 2\cos 1000\pi t + \cos 2000\pi t$; (iii) $m(t) = (\cos 1000\pi t) \times (\cos 3000\pi t)$. For each one, do the following.

(a) Sketch the spectrum of $m(t)$.

(b) Sketch the spectrum of the DSB-SC signal $m(t)\cos(10,000\pi t)$.

[Lathi and Ding, 2009, Q4.2-1]

Recall that the spectrum of $\cos(2\pi f_0 t)$ is given by

Recall that the spectrum of $m(t)\cos(2\pi f_c t)$ is given by

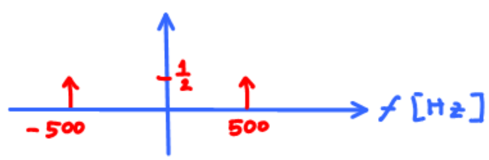
$\frac{1}{2}M(f-f_c) + \frac{1}{2}M(f+f_c)$

shift $M(f)$ to the left by f_c

shift $M(f)$ to the right by f_c.

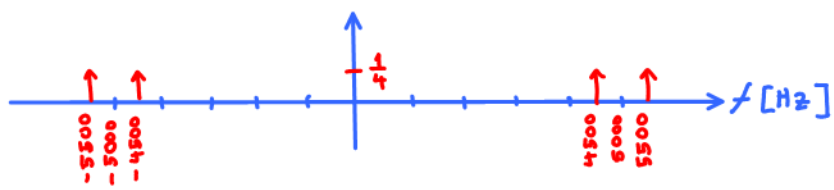
(i.a) $m(t) = \cos 1000\pi t = \cos(2\pi\, 500 t)$

$\hookrightarrow f_0 = 500$ Hz



Here, $f_c = 5000$Hz

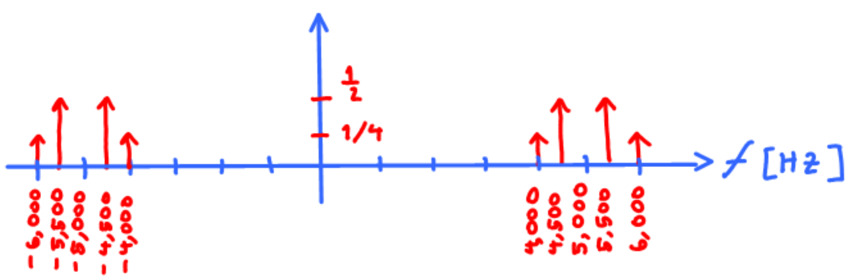
(i.b)



(ii.a) $m(t) = 2\cos(1000\pi t) + \cos(2000\pi t)$

$= 2\cos(2\pi \underbrace{500}_{f_1} t) + \cos(2\pi \underbrace{1000}_{f_2} t)$

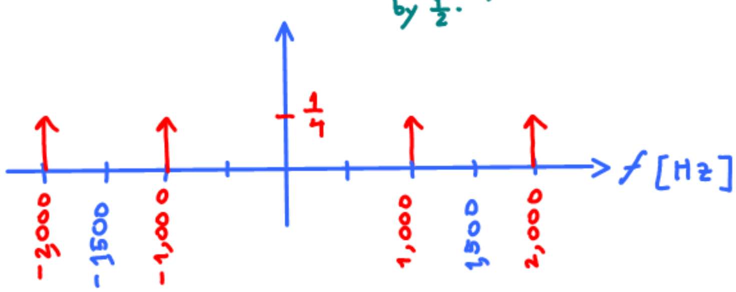
(ii.b)

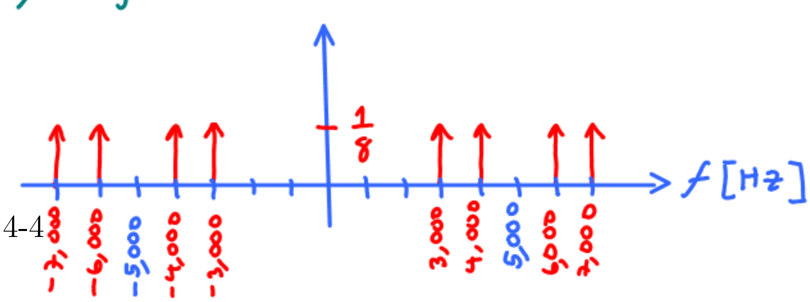


(iii.a) $m(t) = \underbrace{\cos(1000\pi t)}_{\text{part (i.a)}} \times \underbrace{\cos(3000\pi t)}_{= \cos(2\pi(1500)t)}$

from (i.b), multiplication by $\cos(2\pi(1500)t)$ in the time domain is the same as shifting the spectrum content by ± 1500 Hz and vertically scaling by $\frac{1}{2}$.

(iii.b) Here, we multiply again by $\cos(2\pi\, 5{,}000\, t)$. So, the spectrum from (iii.a) is shifted by $\pm 5{,}000$Hz and scaled vertically by $\frac{1}{2}$.





Problem 4. Given a system with input-output relationship of

$$y(t) = 2x(t) + 10,$$

is this system linear? [Carlson and Crilly, 2009, Q2.3-10]

One requirement for a system to be linear is that "proportional changes in the input should give the same proportional changes in the output"

In particular, if $x = 1$ corresponds to $y = 12$, then $x = 1 \times 2$ should correspond to $y = 12 \times 2 = 24$. (Doubling the input causes the output to double.)

In our case, we have $y = 2x + 10$. So, if $x = 1$, $y = 2 \times 1 + 10 = 12$.

For linear system, when $x = 2$, we expect y to be 24.

However, by its definition, when $x = 2$, our system gives $y = 2 \times 2 + 10 = 14 \neq 24$.

Therefore, the system is not linear.

Problem 5. Signal $x(t) = 10\cos(2\pi \times \underbrace{7 \times 10^6}_{f_c} \times t)$ is transmitted to some destination. The received signal is $y(t) = 10\cos(2\pi \times 7 \times 10^6 \times t - \underbrace{\pi/6}_{\theta})$.

We assume that the delay is caused by the propagation time of the signal.

(a) What is the minimum distance between the source and destination?

$$y(t) = 10\cos(2\pi f_c t - \theta) = 10\cos\left(2\pi f_c\left(t - \left(\frac{\theta}{2\pi f_c}\right)\right)\right)$$

In the lecture, we use τ to denote propagation delay.

The amount of time delay can be calculated from $\text{delay} = \frac{\text{distance}}{c}$ ← speed of light.

Therefore, "one possible" distance value is

$$\text{distance} = c \times \text{delay} = c \times \frac{\theta}{2\pi f_c} = \lambda_c \frac{\theta}{2\pi} = 3 \times 10^8 \times \frac{\pi/6}{2\pi \times 7 \times 10^6} = \frac{100}{28} \approx 3.57 \text{ m}$$

wavelength of the carrier

Because the cosine function is periodic, the calculation above gives only one of the many possible distance values. See the discussion in the next part for the proof that 3.57 is the minimum distance.

(b) What are the other possible distances?

By periodicity of cosine,

$$\cos(2\pi f_c t - \theta) = \cos(2\pi f_c t - \theta + 2\pi k) \quad \text{for any integer } k.$$

So, in part (a), we should have considered

$$\cos(2\pi f_c t - \theta + 2\pi k) = \cos\left(2\pi f_c\left(t - \underbrace{\left(\frac{\theta}{2\pi f_c} - \frac{k}{f}\right)}_{\tau}\right)\right) \Rightarrow \text{distance} = c \times \tau = \frac{c}{f_c}\left(\frac{\theta}{2\pi} - k\right) = \lambda_c\left(\frac{\theta}{2\pi} - k\right)$$

This "c" is a subscript to emphasize that this is the wavelength of the carrier

Distance is a positive quantity. So, we need $k < \frac{\theta}{2\pi} = \frac{\pi/6}{2\pi} = \frac{1}{12}$.

In other words, k can be $0, -1, -2, -3, \ldots$

The greater the value of k, the smaller corresponding distance value. Here, "0" is the largest value for k. Therefore, the minimum distance can be found by plugging-in k = 0. The resulting distance is the same as what we (naively) found in part (a).

[Carlson and Crilly, 2009, Q2.3-14]

Other possible values of the distance are $d = \frac{c}{f_c}\left(\frac{\theta}{2\pi} - k\right) \overset{n=-k}{=} \frac{c}{f_c}\left(\frac{\theta}{2\pi} + n\right) = 3.57 + 42.86n$ where $n = 1, 2, 3, \ldots$

$k = -1, -2, -3, \ldots$ $n = 1, 2, 3, \ldots$

From the Fourier transform properties reviewed in lecture, we have seen several interesting integrations.
In particular,

$$\int_{-\infty}^{\infty} G(f)\,df \overset{\star}{=} g(0), \quad \int_{-\infty}^{\infty} g(t)\,dt \overset{\star\star}{=} G(0), \quad \int_{-\infty}^{\infty} x(t)\,y^*(t)\,dt \overset{\star\star\star}{=} \int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df.$$

In this question, we use them to evaluate integrals involving sinc functions. Note that direct integration of a sinc function is difficult. However, its Fourier transform is a simple rectangular function which is easy to evaluate or integrate.

Problem 6 (M2011)**.** Use properties of Fourier transform to evaluate the following integrals. (Do not integrate directly. Recall that $\text{sinc}(x) = \frac{\sin(x)}{x}$.) Clearly state the property or properties that you use. First note that

Note that although "x" is used as the name of the variable here, we can easily change it to "t" or "f".

(a) $\int_{-\infty}^{\infty} 1\,\text{sinc}\left(\sqrt{5}x\right)dx$

① $= 2\pi \frac{\sqrt{5}}{2\pi} x$

② ↳ "freq" $= \frac{\sqrt{5}}{2\pi}$ ⇒ ③ "period" $= \frac{2\pi}{\sqrt{5}}$

④ First zero intercept $= \frac{1}{2} \times$ period $= \frac{\pi}{\sqrt{5}}$

⑧ 1

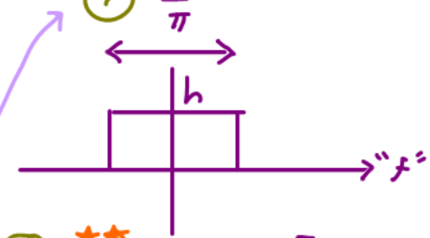
$\frac{2\pi}{\sqrt{5}}$ ③

⑥ $= \frac{1}{\text{width}}$ ⇒ width $= \frac{\sqrt{5}}{\pi}$ ⑦

of the rect. func. in another domain

In another domain,

⑦ $\frac{\sqrt{5}}{\pi}$



⑨ $1 \overset{\star\star}{=}$ Area $= h \times \frac{\sqrt{5}}{\pi}$

$h = \frac{\pi}{\sqrt{5}}$

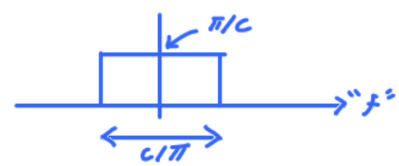
⑩ $\int_{-\infty}^{\infty} \text{sinc}\left(\sqrt{5}x\right)dx \overset{\star}{=} h = \frac{\pi}{\sqrt{5}}$

(b) $\int_{-\infty}^{\infty} \text{sinc}\left(\sqrt{5}x\right)\text{sinc}^*\left(\sqrt{7}x\right)dx$

The function is real-valued. Therefore, we can freely add the conjugate operator.

From part (a), replacing the number $\sqrt{5}$ by an arbitrary constant c, we have

$\text{sinc}(cx) \xrightarrow{\mathcal{F}}$



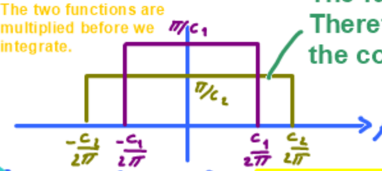
★★★

Therefore,

Here, $c_1 = \sqrt{5}$ and $c_2 = \sqrt{7}$.
So, in our drawing, we will assume $c_2 > c_1$

The two functions are multiplied before we integrate.

$$\int_{-\infty}^{\infty} \text{sinc}(c_1 x)\,\text{sinc}^*(c_2 x)\,dx \overset{\star\star\star}{=} \int_{-\infty}^{\infty}$$



df

The function is real-valued. Therefore, we can freely remove the conjugate operator.

$$= \frac{\pi^{\cancel{2}}}{c_1 c_2} \times \frac{\min\{c_1, c_2\}}{\cancel{\pi}} = \boxed{\frac{\pi}{c_1 c_2} \min\{c_1, c_2\}.}$$

$c_1 = \sqrt{5}$, $c_2 = \sqrt{7}$ ⟶ $= \frac{\pi}{\cancel{\sqrt{5}}\sqrt{7}} \cancel{\sqrt{5}} = \frac{\pi}{\sqrt{7}}$

Interestingly, this is the same answer as $\int_{-\infty}^{\infty} \text{sinc}\left(\sqrt{7}x\right)dx$.

(c) (Optional) $\int_{-\infty}^{\infty} e^{-2\pi f \times 2j} 2\text{sinc}(2\pi f)\left(e^{-2\pi f \times 5j} 2\text{sinc}(2\pi f)\right)^* df$

(d) (Optional) $\int_{-\infty}^{\infty} \text{sinc}\left(\pi(x-5)\right)\text{sinc}\left(\pi\left(x - \frac{7}{2}\right)\right)dx$

(c) The integral is already of the form $\int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$ where

$$X(f) = e^{-2\pi f \times 2j}\, 2\,\mathrm{sinc}(2\pi f) = e^{-j2\pi f(2)}\, G(f)$$

$$Y(f) = e^{-2\pi f \times 5j}\, 2\,\mathrm{sinc}(2\pi f) = e^{-j2\pi f(5)}\, G(f)$$

$$G(f) = 2\,\mathrm{sinc}(2\pi f)$$

From ★★★ (Parseval's theorem), we know that we can evaluate the integral from $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$. So, we will first find $x(t)$ and $y(t)$.

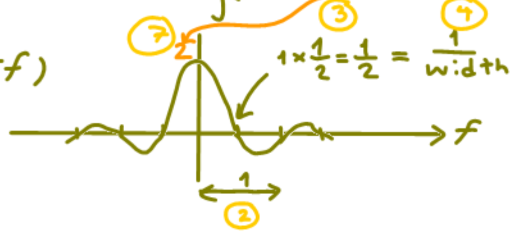
By the time-shifting property, we know that

$$X(f) = e^{-j2\pi f(2)}\, G(f) \xrightarrow{\mathcal{F}^{-1}} x(t) = g(t-2) \quad \text{and}$$

$$Y(f) = e^{-j2\pi f(5)}\, G(f) \xrightarrow{\mathcal{F}^{-1}} y(t) = g(t-5).$$

So, we must first find $g(t)$.

$G(f) = \mathrm{sinc}(2\pi f)$

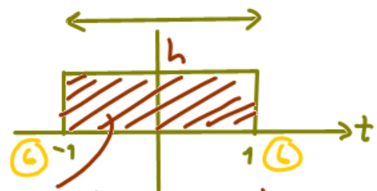


① "freq" = 1

② "period" = $\frac{1}{1}$ = 1

$\xrightarrow{\mathcal{F}^{-1}}$

width = $1/\tfrac{1}{2} = 2$



area = $2 \times h = 2 \Rightarrow h = 1$.

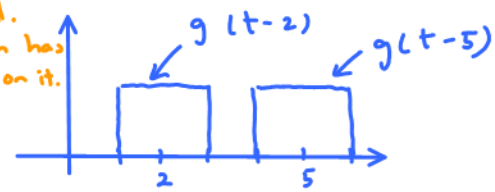
Therefore,

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} g(t-2)\, g^*(t-5)\, dt$$

$g(t)$ is real-valued. So, conjugation has no effect on it.

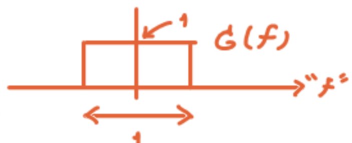
The non-zero parts of these functions do not overlap. Therefore, their product $\equiv 0$.

$$= \int_{-\infty}^{\infty} 0\, dt = 0$$



(d) Start with $g(x) = \mathrm{sinc}(\pi x) \xrightarrow{\mathcal{F}}$

★★★★ (with c = 1)



Next, recall that

$$\mathrm{sinc}(\pi(x - x_0)) \xrightarrow{\mathcal{F}} e^{-j2\pi x_0 f}\, G(f) \quad \text{by the "time"-shifting property.}$$

By ★★★ (Parseval's theorem), the integral under consideration is the same as

$$\int_{-\infty}^{\infty} e^{-j2\pi x_1 f}\, G(f) \left(e^{-j2\pi x_2 f}\, G(f)\right)^* df = \int_{-\infty}^{\infty} |G(f)|^2\, e^{j2\pi(x_2 - x_1)f}\, df$$

same for the G(f) above

$$= \int_{-\infty}^{\infty} G(f)\, e^{j2\pi(x_2 - x_1)f}\, df$$

Note that the last integral here is exactly the same as the inverse Fourier transform of $G(f)$ evaluated at $x = x_2 - x_1$.

$$= \mathrm{sinc}(\pi(x_2 - x_1))$$

Here, $x_2 - x_1 = \frac{7}{2} - 5 = -\frac{3}{2}$. So, the integral is $\dfrac{\sin(-\frac{3}{2}\pi)}{-\frac{3}{2}\pi} = \dfrac{1}{-\frac{3}{2}\pi} = -\dfrac{2}{3\pi}$.

(If $x_2 - x_1$ is an integer, then the integral is 0.)