ECS 332: Principles of Communications **2019/1**

HW 3 — Due: September 13, 4 PM **Solution**

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Instructions

(a) This assignment has 7 pages. Some MATLAB scripts are available from `http://www2.siit.tu.ac.th/prapun/ecs332/ECS332_2019_HW_3_MATLAB.zip`.

(b) (1 pt) Hard-copies are distributed in class. Original pdf file can be downloaded from the course website. Work and write your answers **directly on the provided hardcopy/file** (not on other blank sheet(s) of paper).

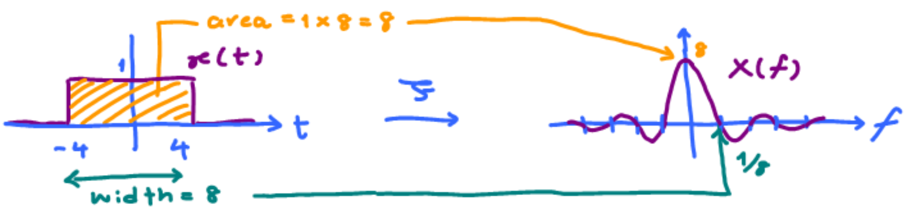
(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

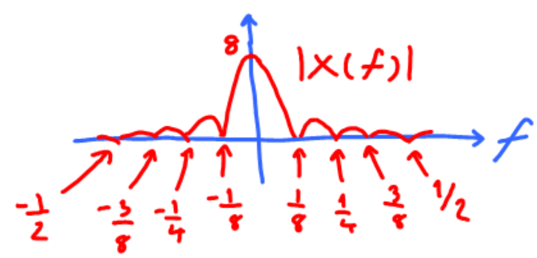
(d) (8 pt) Try to solve all non-optional problems.

(e) Late submission will be heavily penalized.

Problem 1. All of the signals under consideration here are rectangular pulses in the time domain. We know that the Fourier transform of an even rectangular pulse is a sinc function in the freq. domain.

(a) Plot (by hand) the *amplitude spectrum* of the signal $x(t) = \begin{cases} 1, & -4 < t < 4, \\ 0, & \text{otherwise.} \end{cases}$

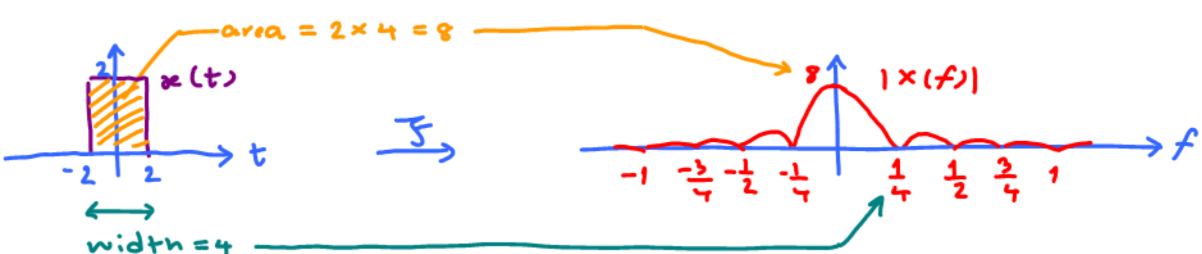




Note that we are not done yet. We are asked to plot the amplitude spectrum (|X(f)|) and not the Fourier transform X(f) itself. Because x(f) is real-valued, |X(f)| is simply the absolute value of X(f). So, the "negative" part of X(f), gets rectified (flipped up).

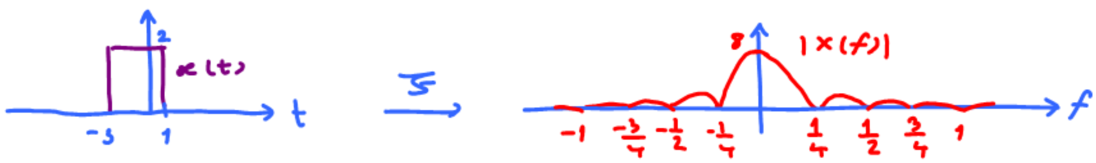
(b) Plot (by hand) the *amplitude spectrum* of the signal $x(t) = \begin{cases} 2, & -2 < t < 2, \\ 0, & \text{otherwise.} \end{cases}$

Here, we simply follow the same technique that was applied in part (a)



(c) Plot (by hand) the *amplitude spectrum* of the signal $x(t) = \begin{cases} 2, & -3 < t < 1, \\ 0, & \text{otherwise.} \end{cases}$

Note that the signal x(t) in this part is the same aa the one in part (b) but shifted to the left. We have seen in class that time-shifting of the whole signal does not change its amplitude spectrum. Therefore, we can simply copy the plot of |X(f)| from part (b) here.



Problem 2. [1] Using MATLAB to find the (amplitude) spectrum[2] of a signal:

A signal $g(t)$ can often be expressed in analytical form as a function of time t, and the Fourier transform is defined as the integral of $g(t)\exp(-j2\pi ft)$. Often however, there is no analytical expression for a signal, that is, there is no (known) equation that represents the value of the signal over time. Instead, the signal is defined by measurements of some physical process. For instance, the signal might be the waveform at the input to the receiver, the output of a linear filter, or a sound waveform encoded as an mp3 file.

In all these cases, it is not possible to find the spectrum by analytically performing a Fourier transform. Rather, the discrete Fourier transform (or DFT, and its cousin, the more rapidly computable fast Fourier transform, or FFT) can be used to find the spectrum or frequency content of a measured signal. The MATLAB function `plotspect.m`, which plots the spectrum of a signal can be downloaded from our course website. Its help portion[3] notes

```
% plotspect(x,t) plots the spectrum of the signal x
% whose values are sampled at time (in seconds) specified in t
```

(a) The function `plotspect.m` should be straightforward to use. For instance, the spectrum of a rectangular pulse[4] $g(t) = 1[0 \le t \le 2]$ can be found using:

[1]Based on [Johnson, Sethares, and Klein, 2011, Sec 3.1 and Q3.3].

[2]also referred to by "amplitude spectrum" or simply "spectrum"

[3]You can view the "help" portion for a MATLAB function `xxx` by typing `help xxx` at the MATLAB prompt. If you get an error such as `xxx` not found, then this means either that the function does not exist, or that it needs to be moved into MATLAB's search path.

[4]Here, we define a rectangular pulse using the indicator function $1[\cdot]$. This function outputs a 1 when the statement inside the square brackets is true; otherwise, it outputs a 0. For example,

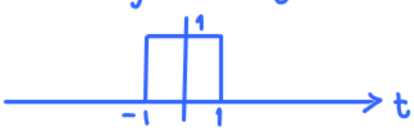
$$1[0 \le t \le 2] = \begin{cases} 1, & 0 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

```
% specrect.m plot the spectrum of a square wave
close all
Ts=1/100;            % time interval between adjacent samples
t=0:Ts:20;           % create a time vector
x=[t <= 2];          % rectangular pulse $1[0 \leq t \leq 2]$
plotspect(x,t)       % call plotspect to draw spectrum
xlim([-5,5])         % look only from f = -5 to f = 5 Hz
```

The output of `specrect.m` is shown in Figure 3.1. The top plot shows the first 20 seconds of $g(t)$. The bottom plot shows $|G(f)|$.

Let

$x(t) = g(t+1) = g(t-(-1))$



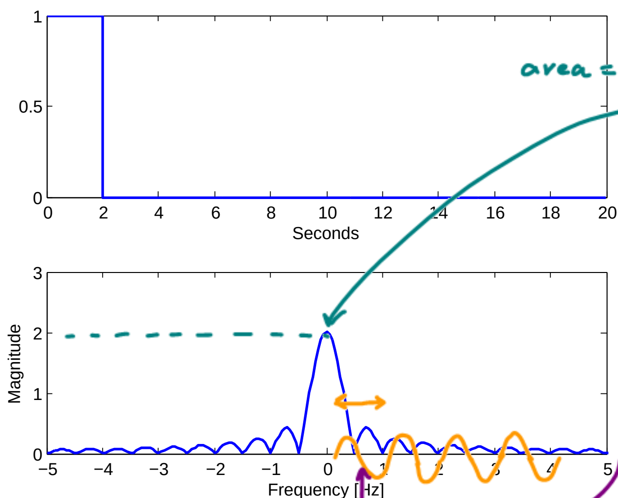
Then, by the time-shift property,

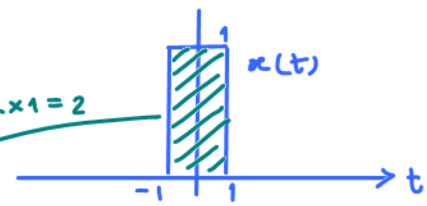
$X(f) = e^{j2\pi f(-(-1))} G(f)$

and

$|X(f)| = |G(f)|$

$|e^{j\theta}| \equiv 1$ for any θ





area = 2×1 = 2

width = 2

"period" = $\frac{1}{2} \times 2 = 1$

"freq." = $\frac{1}{1} = 1$

$X(f) = 2\,\mathrm{sinc}(2\pi 1 f)$

$\frac{1}{2} = \frac{1}{\text{width}}$

Figure 3.1: Plots from `specrect.m`

(i) Use what we studies in class about the Fourier transform of a rectangular pulse (and the time-shift property) to find a simplified expression for $|G(f)|$.

$$|G(f)| = |X(f)| = 2|\mathrm{sinc}(2\pi f)|$$

(ii) Use `MATLAB` to plot your analytical expression derived in part (i). Did your plot agree with the lower plot in Figure 3.1? Attach the printed plot on another page (or, for online submission, add one more blank page to the HW file and then copy and paste the generated plot); write the page number as page 3-8. (Remark: There will be more plots to be put on this page.)

Caution: The built-in `sinc` function in `MATLAB` is defined using the normalized version. So, you will need to remove a factor of π from the argument of each sinc function found in part (i) when you type it into `MATLAB`.

The two plots should be exactly the same.

(b) Now consider an exponential pulse

$$s(t) = e^{-t}u(t) \text{ where } u(t) \text{ is the unit step function.}$$

(i) Modify the code in `specrect.m` to show the (magnitude) spectrum $|S(f)|$. Include the printed plot on page 3-8 (or, for online submission, copy and paste the generated plot to page 3-8).

(ii) Find $S(f)$ and $|S(f)|$ analytically. (Hopefully, you still remember how to integrate exponential function.)

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(1+j2\pi f)t}dt$$

direct FT formula

$$= \frac{1}{-(1+j2\pi f)} e^{-(1+j2\pi f)t}\Big|_{t=0}^{\infty} = -\frac{1}{1+j2\pi f}(0-1) = \frac{1}{1+j2\pi f}$$

Recall that the magnitude of a complex number $z = x+jy$ is $\sqrt{x^2+y^2}$ and that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

$$|S(f)| = \frac{1}{\sqrt{1+(2\pi f)^2}}$$

(iii) Plot your analytical expression in part (ii) and compare with the plot in part (i). Include the printed plot on page 3-8 (or, for online submission, copy and paste the generated plot to page 3-8).

(iv) `MATLAB` can also perform symbolic manipulation when symbolic toolbox is installed. It can find the Fourier transform of a symbolic expression via the command `fourier`. Unfortunately, the `fourier` command use the ω-version of the defiition. So, to convert the answer to the f-version, we also need to substitute $\omega = 2\pi f$. This is done automatically in our provided function `fourierf`.

Run the file `SymbFourier.m`. Check whether you have the same result as part (ii).

Here is the result displayed on the command window:

```
>> SymbFourier
S =
1/(a + pi*f*2*i)
```


Setting the value of the variable "a" to 1, we have same result as in (i).

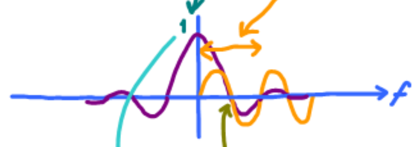
Problem 3.

(a) Suppose the Fourier transform of a signal $x(t)$ is given by

$$X(f) = \operatorname{sinc}(5\pi f) = \frac{\sin(5\pi f)}{(5\pi f)}. \overset{①}{=} \frac{\sin\left(2\pi \frac{5}{2} f\right)}{5\pi f}.$$

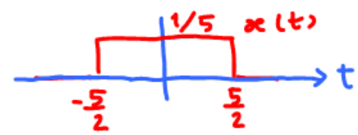
(i) Plot (by hand) $x(t)$. ⑤

"period" $= \frac{2}{5}$ ③ $\Leftarrow$ "freq" $= \frac{5}{2}$ ②



④ $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5} = \frac{1}{\text{width of } \sqcap}$

So, the rectangular pulse in the time domain has width = 5.
Its area under the graph must be 1. So, its height must be 1/5.



(ii) Find $\int_{-\infty}^{\infty} X(f)df$. (Hint: This integration is exactly the inverse Fourier transform formula with $t = 0$.)

$$\int_{-\infty}^{\infty} X(f)\,df = x(0) = \frac{1}{5}.$$

(b) Suppose the Fourier transform of a signal $y(t)$ is given by

$$Y(f) = \operatorname{sinc}^2(5\pi f) = \left(\frac{\sin(5\pi f)}{(5\pi f)}\right)^2.$$

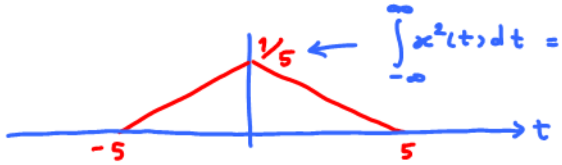
(i) Plot (by hand) $y(t)$.

Note that $Y(f) = (X(f))^2 = X(f) \times X(f)$. ← multiplication

By the convoltuion property of Fourier transform, we know that

$y(t) = x(t) * x(t)$ ← convolution. ← We discussed convolution of rect. functions in class.

$\int_{-\infty}^{\infty} x^2(t)\,dt = \left(\frac{1}{5}\right)^2 \times 5 = \frac{1}{5}$



(ii) Find $\int_{-\infty}^{\infty} Y(f)df$. $= y(0) = \frac{1}{5}$.

Alternatively, one can use the Parseval's theorem:

$$\int_{-\infty}^{\infty} Y(f)\,df = \int_{-\infty}^{\infty} X^2(f)\,df = \int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{5}.$$

Note that we don't have to write $|\cdot|$ because $x(t)$ and $X(f)$ are real-valued.

Alternatively, we can try to solve (a.i) and (b.i) via formula.

(a.i) We know that

$$2a\,\text{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} 1[|t| \le a] \leftarrow \text{shown in class.}$$

Therefore, $\text{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a]$.

Here, $2a = 5$. So, $a = 5/2$.

(b.i)

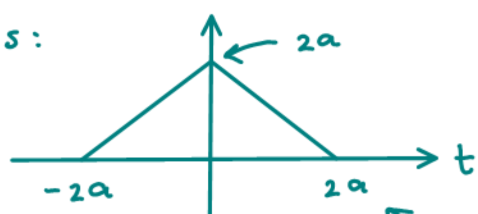
$$\text{sinc}^2(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a] * \frac{1}{2a} 1[|t| \le a]$$

$$= \frac{1}{4a^2}\left(1[|t| \le a] * 1[|t| \le a]\right)$$

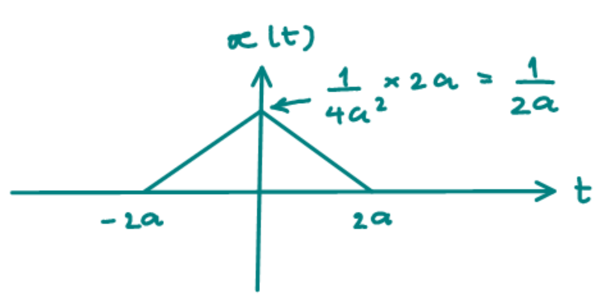
So, we can solve this question if we can find the convolution of $1[|t| \le a]$ with itself.

This is also discussed in class:

$$1[|t| \le a] * 1[|t| \le a] =$$



Therefore, the plot of $x(t)$ should be the same as but scaled vertically by a factor of $1/4a^2$:



Problem 4. The Fourier transform of the triangular pulse $g(t)$ in Figure 3.2a is given as

$$G(f) = \frac{1}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right)$$

Using this information, and the time-shifting and time-scaling properties, find the Fourier transforms of the signals shown in Figure 3.2b, c, d, e, and f.

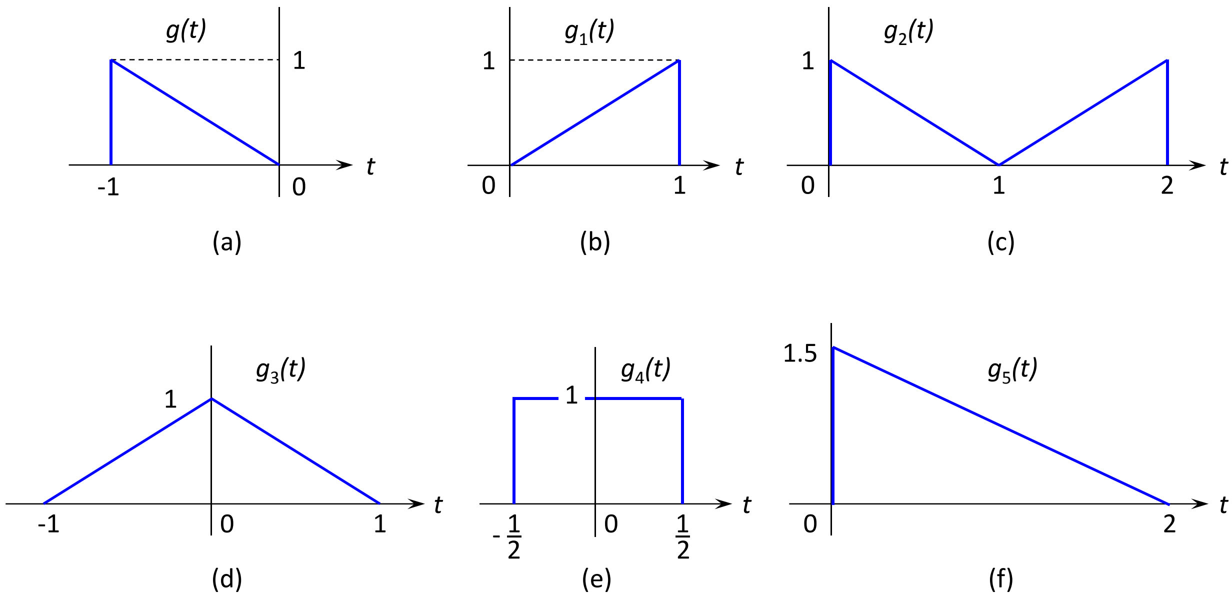


Figure 3.2: Problem 4

Remark: Don't forget to simplify your answers. For example, the answer in part (d) should be of the form $\text{sinc}^2(\cdot)$ and the answer in part (e) should be of the form $\text{sinc}(\cdot)$

(b) Note that $g_1(t) = g(-t)$.

Recall that $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$.

Here, $a = -1$.

Therefore, $G_1(f) = \frac{1}{|-1|} G\left(\frac{f}{-1}\right) = \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right)$

(c) Note that $g_2(t) = g(t-1) + g_1(t-1)$

$$\Rightarrow \quad G_2(f) = e^{-j2\pi f} G(f) + e^{-j2\pi f} G_1(f) = \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1 + e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

Here, we write "ω" instead of "2πf". After we're done massaging the expressions, we change "ω" back to "2πf".

$$\frac{e^{-j\omega}}{\omega^2}\left(2\cos(\omega) - j\omega(2j)\sin\omega - 2\right)$$

$$= \frac{2e^{-j2\pi f}}{(2\pi f)^2}\left(\cos(2\pi f) + 2\pi f \sin(2\pi f) - 1\right)$$

(d) Note that $g_3(t) = g(t-1) + g_1(t+1)$

$$\Rightarrow \quad G_3(f) = e^{-j2\pi f} G(f) + e^{j2\pi f} G_1(f) = \frac{1}{\omega^2}\left(1 - \cancel{j\omega} - e^{-j\omega} + 1 + \cancel{j\omega} - e^{j\omega}\right)$$

$$= \frac{1}{\omega^2}\left(2 - e^{-j\omega} - e^{j\omega}\right) = -\frac{1}{\omega^2}\left(e^{j\omega} - 2 + e^{-j\omega}\right) = -\frac{1}{\omega^2}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)^2$$

$$= -\frac{1}{\omega^2}\left(2j\sin\frac{\omega}{2}\right)^2 = \frac{\sin^2\left(\frac{\omega}{2}\right)}{\left(\frac{\omega}{2}\right)^2} = \text{sinc}^2\left(\frac{\omega}{2}\right) = \text{sinc}^2(\pi f)$$

(e) Note that $g_4(t) = g(t-\frac{1}{2}) + g_1(t+\frac{1}{2})$.

$$\Rightarrow \quad G_4(f) = e^{-j\omega/2} G(f) + e^{j\omega/2} G_1(f)$$

use part (b)

$$= e^{-j\omega/2}\frac{1}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right) + e^{j\omega/2}\frac{1}{\omega^2}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{1}{\omega^2}\left(\cancel{e^{j\frac{\omega}{2}}} - j\omega e^{j\frac{\omega}{2}} - \cancel{e^{-j\frac{\omega}{2}}} + \cancel{e^{-j\frac{\omega}{2}}} + j\omega e^{-j\frac{\omega}{2}} - \cancel{e^{j\frac{\omega}{2}}}\right) = \frac{-j}{\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right)$$

$$= \frac{(-j)}{\omega}(2j)\sin(\omega/2) = \frac{\sin(\omega/2)}{\omega/2} = \text{sinc}\left(\frac{\omega}{2}\right) = \text{sinc}(\pi f)$$

(f) Note that $g_5(t) = 1.5\, g\left(\frac{1}{2}(t-2)\right)$

$$\Rightarrow \quad G_5(f) = 1.5 \times \frac{1}{1/2} G\left(\frac{f}{1/2}\right) e^{-j2\omega} = 3G(2f)e^{-j}$$

$$= 3 \times \frac{1}{(2\pi 2f)^2}\left(e^{j2\omega} - j2\omega e^{j2\omega} - 1\right)e^{-j2\omega} = \frac{3}{4\omega^2}\left(1 - 2j\omega - e^{-2j\omega}\right)$$

$$= \frac{3}{4(2\pi f)^2}\left(1 - j4\pi f - e^{-j4\pi f}\right)$$

Extra Question

Here is an optional question for those who want more practice.

Problem 5. Listen to the Fourier's Song (Fouriers_Song.mp3) which can be downloaded from `http://sethares.engr.wisc.edu/mp3s/fourier.html`

Which properties of the Fourier Transform can you recognize from the song? List them here.

Q2.a.ii In the bottom part of Figure (i) below, the theoretical expression in part (i) is plotted using the "o" marks on top of the provided plot from `specrect.m`. The two plots match perfectly.

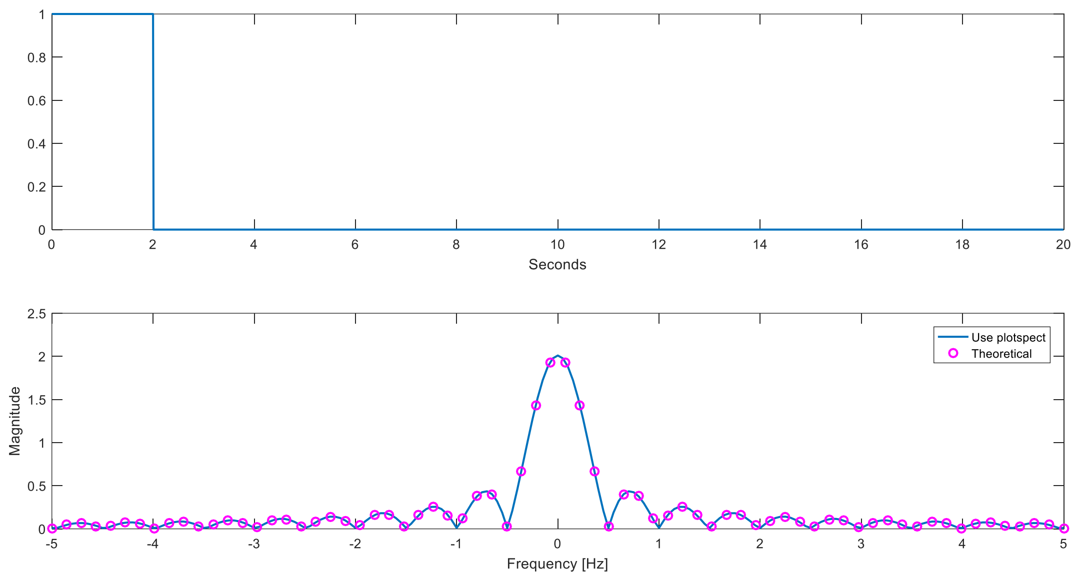


Figure (i)

Q2.b.i The magnitude spectrum plot from the modified `specrect.m` is provided in the bottom part of Figure (ii) below.

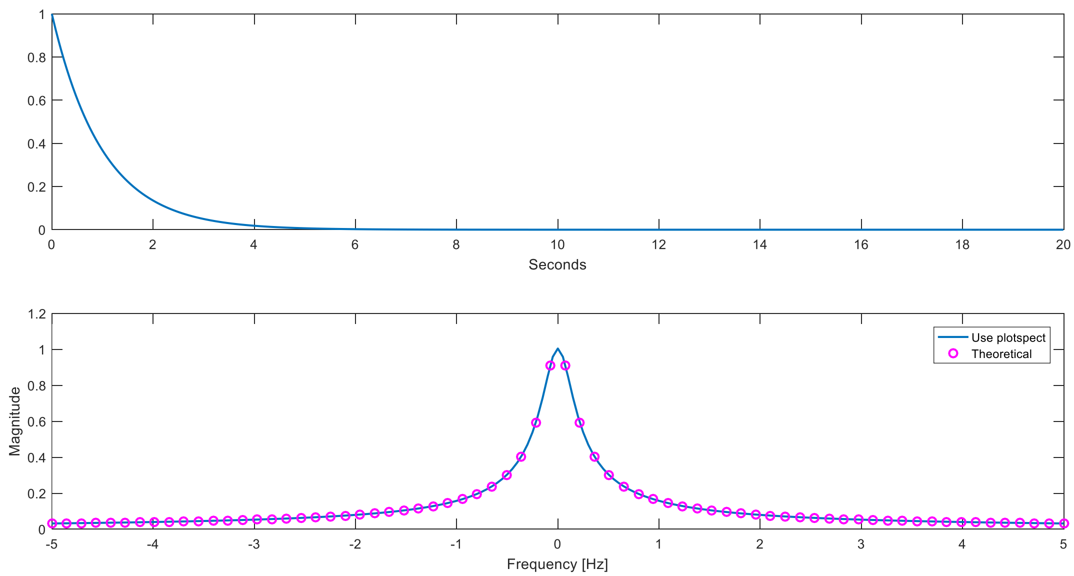


Figure (ii)

Q2.b.iii The analytical expression in part (ii) is plotted using the "o" marks on top of the plot from `specrect.m` from part (i). The two plots match perfectly.