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ECS 332: Principles of Communications **2019/1**

HW 2 — Due: August 30, 4 PM **Solution**

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Instructions

(a) This assignment has 5 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all problems.

(e) Late submission will be heavily penalized.

Problem 1. In class, we have seen how to use the Euler's formula to show that

$$\cos^2 x = \frac{1}{2}\left(\cos(2x) + 1\right).$$

For this question, ***apply similar technique*** to show that

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

$$\cos A \cos B = \frac{1}{2}\left(e^{jA}+e^{-jA}\right)\times\frac{1}{2}\left(e^{jB}+e^{-jB}\right)$$

$$= \left(e^{jA}+e^{-jA}\right)\left(e^{jB}+e^{-jB}\right)\times\frac{1}{4}$$

$$= \Big(\underbrace{e^{j(A+B)}+e^{-j(A+B)}}_{2\cos(A+B)}+\underbrace{e^{j(A-B)}+e^{-j(A-B)}}_{2\cos(A-B)}\Big)\frac{1}{4}$$

$$= \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$$

Steps:

① Replace cos and sin with complex exponential functions.

② Simplify or rearrange the expression

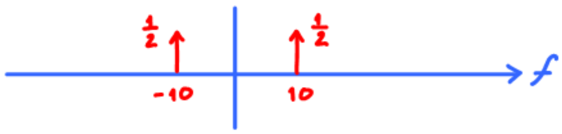
③ Convert back to cos and sin

First, we convert the given expressions into complex exponential functions. Then, we use the fact that $e^{j2\pi f_0 t}$ in the time domain corresponds to the delta function at $f = f_0$ in the frequency domain

Problem 2. Plot (by hand) the Fourier transforms of the following signals

(a) $\cos(\underbrace{20\pi t}_{A = 2\pi(10)t}) = \frac{e^{jA}+e^{-jA}}{2} = \frac{1}{2}e^{jA} + \frac{1}{2}e^{-jA} = \frac{1}{2}e^{j2\pi(10)t} + \frac{1}{2}e^{j2\pi(-10)t}.$

So, the plot of its Fourier transform is



Alternatively, one may simply remember that the Fourier transform of $\cos(2\pi f_0 t)$ is simply delta functions of size $\frac{1}{2}$ at f_0 and $-f_0$.

(b) $\cos(20\pi t) + \cos(40\pi t)$

For $\cos(\underbrace{40\pi t})$, the corresponding frequencies are $\underbrace{\pm 20}$ Hz.

$2\pi f_0 t = 40\pi t$

$f_0 = 20$

So, the plot of the Fourier transform of $\cos(20\pi t) + \cos(40\pi t)$ is

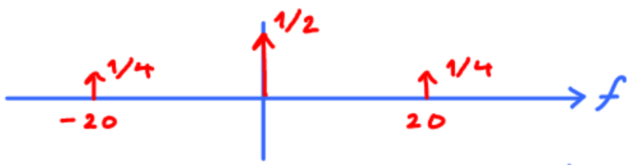
already got from part (a)

(c) $(\cos(20\pi t))^2$

$(\cos(\underbrace{20\pi t}_{A = 20\pi t = 2\pi(10)t}))^2 = (\cos A)^2 = \left(\frac{1}{2}(e^{jA} + e^{-jA})\right)^2 = \frac{1}{4}\left(e^{2jA} + 2 + e^{-2jA}\right)$

$= \frac{1}{4}e^{j2\pi(20)t} + \frac{1}{2}\underbrace{e^{j2\pi(0)t}}_{1} + \frac{1}{4}e^{j2\pi(-20)t}$

So, the plot of its Fourier transform is

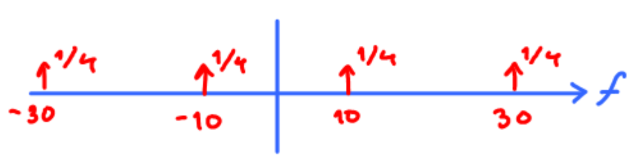


(d) $\cos(\underbrace{20\pi t}_{A = 20\pi t = 2\pi(10)t}) \times \cos(\underbrace{40\pi t}_{B = 40\pi t = 2\pi(20)t}) = \cos(A)\cos(B) = \frac{1}{2}(e^{jA} + e^{-jA})\frac{1}{2}(e^{jB} + e^{-jB})$

$= \frac{1}{4}\left(e^{j(A+B)} + e^{j(-A+B)} + e^{j(A-B)} + e^{j(-A-B)}\right)$

$= \frac{1}{4}\left(e^{j2\pi(30)t} + e^{j2\pi(10)t} + e^{j2\pi(-10)t} + e^{j2\pi(-30)t}\right)$

So, the plot of its Fourier transform is



(e) $(\cos(20\pi t))^2 \times \cos(40\pi t) = \left(\underbrace{\frac{1}{4} e^{j2\pi(20)t} + \frac{1}{2} + \frac{1}{4} e^{j2\pi(-20)t}}_{\text{from part (c)}}\right) \times \left(\frac{1}{2} e^{j2\pi(20)t} + \frac{1}{2} e^{j2\pi(-20)t}\right)$

$$= \frac{1}{8} e^{j2\pi(40)t} + \frac{1}{4} e^{j2\pi(20)t} + \underbrace{\frac{1}{8} e^{0} + \frac{1}{8} e^{0}}_{\frac{1}{4} e^{j2\pi(0)t}} + \frac{1}{4} e^{j2\pi(-20)t} + \frac{1}{8} e^{j2\pi(-40)t}$$

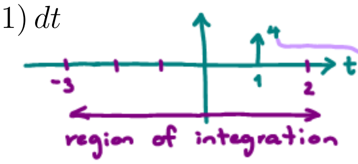
Problem 3. Evaluate the following integrals:

(a) First, recall that $\int_A \delta(t)\,dt = \begin{cases} 1, & 0 \in A, \\ 0, & 0 \notin A. \end{cases}$ In particular, $\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$

(i) $\int\limits_{-\infty}^{\infty} 2\delta(t)\,dt = 2\int_{-\infty}^{\infty} \delta(t)\,dt = 2 \times 1 = 2.$

(ii) $\int\limits_{-3}^{2} 4\delta(t-1)\,dt$

Consider the function $4\delta(t-1)$ graphically.

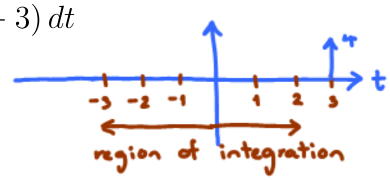


The area under the curve from -3 to 2 includes the arrow area which is 4.

So, $\int_{-3}^{2} 4\delta(t-1)\,dt = 4.$

(iii) $\int\limits_{-3}^{2} 4\delta(t-3)\,dt$

Consider the function $4\delta(t-3)$ graphically.



The area under the curve from -3 to 2 does not include the arrow area.

Therefore, $\int_{-3}^{2} 4\delta(t-3)\,dt = 0.$

(b) $\int\limits_{-\infty}^{\infty} \delta(t)\,\underbrace{e^{-j2\pi ft}}\,dt = \int_{-\infty}^{\infty} g(t)\,\delta(t)\,dt \underset{\text{sifting property}}{=} g(0) = e^{-j2\pi ft}\Big|_{t=0} = e^{0} = 1$

(c)

(i) $\int\limits_{-\infty}^{\infty} \delta(t-2)\,\underbrace{\sin(\pi t)}\,dt = \int_{-\infty}^{\infty} g(t)\,\delta(t-2)\,dt \overset{\text{sifting property v2}}{=} g(2) = \sin(\pi t)\Big|_{t=2} = \sin(2\pi) = 0$

sifting property v2

(ii) $\int\limits_{-\infty}^{\infty} \delta(t+3)\, e^{-t} dt = \int\limits_{-\infty}^{\infty} g(t)\, \delta(t-(-3))dt = g(-3) = e^{-t}\Big|_{t=-3} = e^{-(-3)} = e^{3}$

sifting property v2

(iii) $\int\limits_{-\infty}^{\infty} e^{(x-1)} \cos\left(\frac{\pi}{2}(x-5)\right) \delta(x-3)dx = \int\limits_{-\infty}^{\infty} g(t)\, \delta(x-3)dx = g(3) = e^{x-1}\cos\left(\frac{\pi}{2}(x-5)\right)\Big|_{x=3} = e^{2}\cos(-\pi) = -e^{2}$

Note that the "x" here is just a dummy variable. It takes the role of "t" in our formula.

(d) This part has the delta function in the form $\delta(T-t)$.

We use the "change of variables" technique to evaluate the integral: $\int\limits_{-\infty}^{\infty} g(t)\,\delta(T-t)\,dt = -\int\limits_{\infty}^{-\infty} g(T-\tau)\,\delta(\tau)\,d\tau$

change of variables $\begin{cases}\tau = T-t\\ t = T-\tau\\ dt = -d\tau\end{cases}$ $\begin{cases} t=\tau \\ dt = d\tau\end{cases}$

$= \int\limits_{-\infty}^{\infty} g(T-t)\,\delta(t)\,dt = g(T-0) = g(T).$

(i) $\int\limits_{-\infty}^{\infty} (t^3+4)\, \delta(1-t)dt = t^3+4\Big|_{t=1} = 1^3+4 = 1+4 = 5$

(ii) $\int\limits_{-\infty}^{\infty} g(2-t)\, \delta(3-t)dt = g(2-t)\Big|_{t=3} = g(2-3) = g(-1)$

Remark: From $\int\limits_{-\infty}^{\infty} g(t)\,\delta(T-t)\,dt = g(T)$, we get $\int\limits_{-\infty}^{\infty} g(\tau)\,\delta(t-\tau)\,d\tau = g(t)$ simply by variable renaming $\left(\begin{smallmatrix} t\to\tau \\ T\to t\end{smallmatrix}\right)$

$(g*\delta)(t)$

(e) $\int\limits_{-2}^{2} \delta(2t)\, dt$

$= \int\limits_{-4}^{4} \delta(x)\frac{1}{2}dx = \frac{1}{2}\times\int\limits_{-4}^{4}\delta(x)\,dx = \frac{1}{2}\times 1 = \frac{1}{2}.$

change of variables $\begin{cases} x = 2t \\ t = \frac{1}{2}x \\ dt = \frac{1}{2}dx\end{cases}$

$0 \in [-4,4]$

Alternatively, we know that $\delta(at) = \frac{1}{|a|}\delta(t)$.

Therefore, $\delta(2t) = \frac{1}{2}\delta(t)$.

Hence, $\int\limits_{-2}^{2} \delta(2t)\,dt = \int\limits_{-2}^{2} \frac{1}{2}\delta(t)\,dt$

$= \frac{1}{2}\int\limits_{-2}^{2}\delta(t)\,dt = \frac{1}{2}\times 1 = \frac{1}{2}.$

$0 \in [-2,2]$

Problem 4. Consider the signal $g(t)$ shown in Figure 2.1.

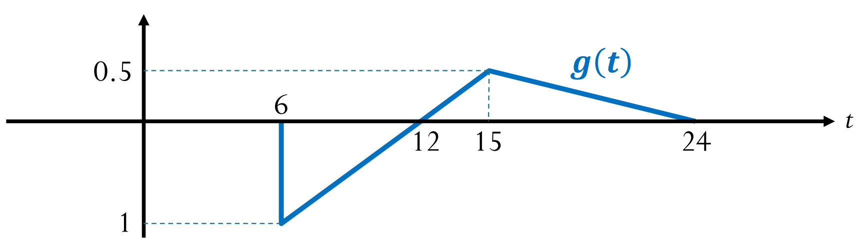


Figure 2.1: Problem 4

(a) Carefully sketch the following signals:

(i) $y_1(t) = g(-t)$

(ii) $y_2(t) = g(t+6)$

(a)

(i) Recall the time inversion (time reversal) operation

$g(-t)$ is the mirror image of $g(t)$ about the vertical axis.

(ii) Recall the time shifting operation:

$g(t-T)$ represents $g(t)$ time-shifted by T.

If T is positive, the shift is to the right (delay).

If T is negative, the shift is to the left (by $|T|$).

Here, $y_2(t) = g(t+6) = g(t-(-6))$.

So, $y_2(t)$ is simply $g(t)$ shifted to the left by 6 time units.

(iii) Recall the time scaling operation:

$g(at)$ is $g(t)$ compressed in time by the factor a.

for $a > 1$

So, $y_3(t) = g(3t)$ is simply $g(t)$ compressed in time by a factor of 3.

(iv) The tricky one would be $g(6-t)$.

There are two ways to think about it

1. $g(t) \xrightarrow{\text{time inversion}} g(-t) \xrightarrow{\text{time shift, } T=6} g(-(t-6))$

mirror image about the vertical axis

shift to the right by 6

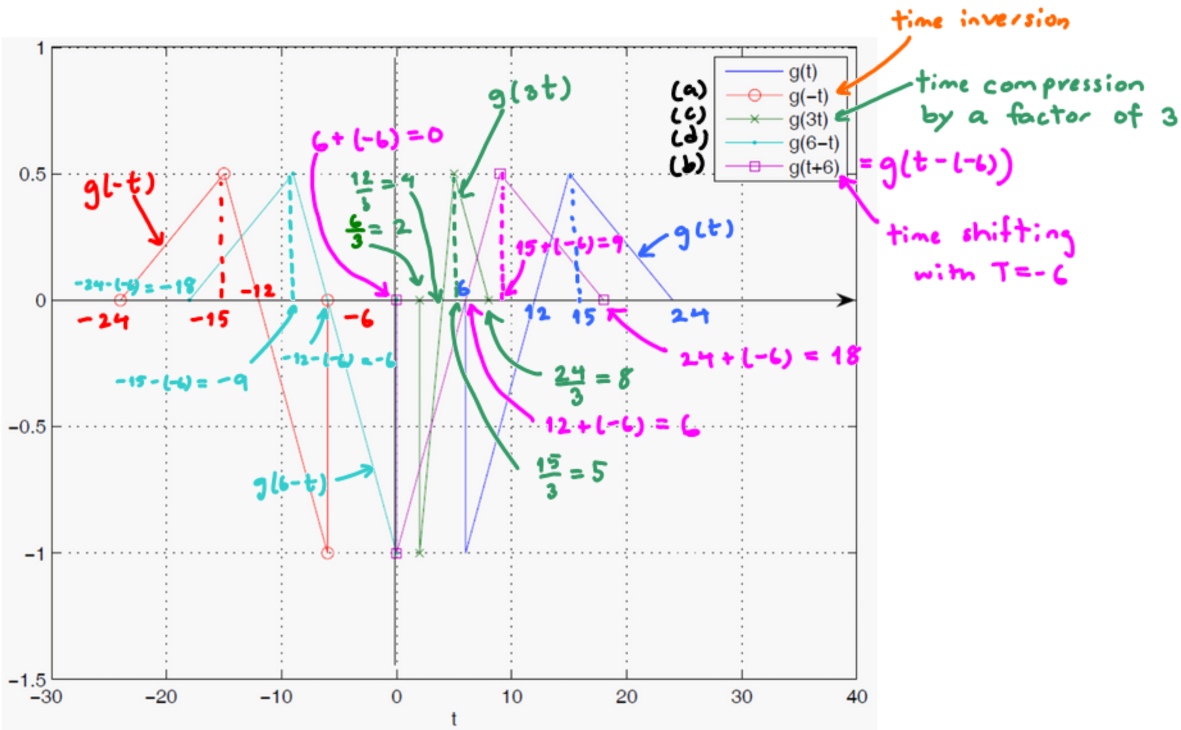
2. $g(t) \xrightarrow{\text{time shift, } T=-6} g(t+6) \xrightarrow{\text{time inversion}} g(-t+6)$

shift to the left by 6

mirror image of $g(t+6)$ about the vertical axis

(iii) $y_3(t) = g(3t)$

(iv) $y_4(t) = g(6 - t)$.



(b) Find the "net" area under the graph for each of the signals in the previous part. (Mathematically, this is equivalent to integrating each signal from $-\infty$ to $+\infty$. However, directly calculating and combining positive and negative areas from the plots should be easier.) First, note that, for any constant m, c,

$$\int_{-\infty}^{\infty} g(mt+c)\,dt = \begin{cases} \int_{-\infty}^{\infty} g(x)\frac{1}{m}dx = \frac{1}{m}\int_{-\infty}^{\infty} g(x)dx, & m>0 \\ \int_{\infty}^{-\infty} g(x)\frac{1}{m}dx = -\frac{1}{m}\int_{-\infty}^{\infty} g(x)dx, & m<0 \end{cases} = \frac{1}{|m|}\int_{-\infty}^{\infty} g(x)dx$$

$x = mt+c$

$dx = m\,dt$

$dt = \frac{1}{m}dx$

Now, for us, $\int_{-\infty}^{\infty} g(t)\,dt = -\underbrace{\left(\frac{1}{2}\times 1\times \overset{12-6}{6}\right)}_{\text{area under the first triangle}} + \left(\frac{1}{2}\times\frac{1}{2}\times \overset{24-12}{12}\right) = -3+3 = 0.$

Therefore, $\int_{-\infty}^{\infty} g(mt+c)\,dt = 0$ for any m, c.

Note:

	m	c
(i)	-1	0
(ii)	1	6
(iii)	0	0
(iv)	-1	6