

HW 7 — Due: Not Due **Solution**

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**Problem 1.** Find the “perceived” frequency when we sample the following signals at sampling rate  $f_s = 12$  [Sa/s]:

Parts a and b consider cosine with no phase shift. Therefore, we will apply the folding technique.

(a) the signal  $\cos(2\pi(1,111,111)t)$

$f_o = 1,111,111$   
 Remainder =  $f_o - f_s \lfloor \frac{f_o}{f_s} \rfloor$   
 $= 7 \leftarrow \text{still} > \frac{f_s}{2}$   
 $f_r = \frac{f_s}{2} - 1 = 6 - 1 = 5 \text{ Hz}$

Alternatively,  $12 \overline{) 1,111,111}$

92592
108
31
24
71
60
111
108
31
24
3

(b) the signal  $\cos(2\pi(111,111)t)$

$f_o = 111,111$   
 Remainder =  $f_o - f_s \lfloor \frac{f_o}{f_s} \rfloor$   
 $= 3$   
 $f_r = 3 \text{ Hz}$

Alternatively,  $12 \overline{) 111,111}$

9259
108
31
24
71
60
111
108
3

Parts c and d consider complex expo. Therefore, we will apply the tunneling technique.

(c) the signal  $e^{j2\pi(11,111)t}$

$f_o = 11,111$   
 Remainder =  $f_o - f_s \lfloor \frac{f_o}{f_s} \rfloor$   
 $= 11$   
 $f_r = -1 \text{ Hz}$

(d) the signal  $e^{-j2\pi(1,111)t}$

$f_o = -1,111$   
 Remainder = 7  
 $f_r = 5 \text{ Hz}$

## Derivation of the "folding" and the "tunneling" techniques

Finding the "perceived" frequency of  $g(t) = e^{j2\pi f_0 t}$  when the sampling rate is  $f_s$ :

Method 1: Analysis via the reconstruction equation.

First, note that  $g(t) = e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} G(f) = \delta(f - f_0)$

With the sampling rate  $= f_s$ , we know that

$$G_s(f) = \sum_k f_s G(f - kf_s) = \sum_k f_s \delta(f - kf_s - f_0) = f_s \sum_k \delta(f - (f_0 + kf_s))$$

↑ Recall that this is periodic with period  $f_s$ .

Now, the reconstruction equation gives

$$G_r(f) = \text{LPF}\{G_s(f)\} \text{ where } H_{\text{LP}}(f) = \begin{cases} T_s, & -\frac{f_s}{2} < f \leq \frac{f_s}{2} \\ 0, & \text{otherwise.} \end{cases}$$

(or, equivalently,  $g_r(t) = \text{LPF}\{g_s(t)\}$ )

Therefore, only the parts of  $G_s(f)$  that are between  $-\frac{f_s}{2} < f \leq \frac{f_s}{2}$  will survive the LPF (and will also be further scaled by  $T_s$ ).

Our task now is then to find all value(s) of integer  $k$  such that

$$-\frac{f_s}{2} < f_0 + kf_s \leq \frac{f_s}{2}$$

$$-\frac{1}{2} - \frac{f_0}{f_s} < k \leq \frac{1}{2} - \frac{f_0}{f_s}$$

Note that the difference between these two numbers is one. So, there is exactly one value of  $k$  that satisfies such condition.

$$\text{so, } k = \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor$$

Note that " $\lfloor \cdot \rfloor$ " is the "floor" function.

One may also consider  $k = \left\lceil -\frac{1}{2} - \frac{f_0}{f_s} \right\rceil$ . However, because the equality in the condition has equality at the upper-bound, the ceiling function of the lower bound will give the wrong answer when the lower bound is an integer itself.

This gives  $G_r(f) = T_s f_s \delta(f - (f_0 + kf_s))$  where  $k = \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor$

the only term in  $G_s(f)$  that is in the passband of the LPF

$$= \delta(f - (f_0 + kf_s))$$

$$\downarrow \mathcal{F}^{-1}$$

$$g_r(t) = e^{j2\pi (f_0 + kf_s) t} \text{ where } k = \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor.$$

Hence, the "perceived" frequency is  $f_v = f_0 + kf_s = f_0 + f_s \left\lfloor \frac{1}{2} - \frac{f_0}{f_s} \right\rfloor$ .

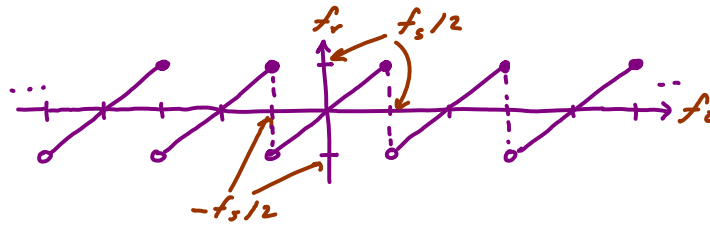
In part (c),  $f_0 = 11,111 \Rightarrow k = \left\lfloor \frac{1}{2} - \frac{11,111}{12} \right\rfloor = \left\lfloor -925.41 \right\rfloor = -926$

$$\Rightarrow f_v = 11,111 + (-926) \times 12 = -1 \text{ Hz.}$$

In part (d),  $f_0 = -1,111 \Rightarrow k = \left\lfloor \frac{1}{2} + \frac{1,111}{12} \right\rfloor = \left\lfloor 93.1 \right\rfloor = 93$

$$\Rightarrow f_v = -1,111 + 93 \times 12 = 5 \text{ Hz.}$$

Remark: The plot of  $f_r = f_o + f_s \lfloor \frac{1}{2} - \frac{f_o}{f_s} \rfloor$  is shown below:



Observe the "tunneling effect":

- i)  $f_r$  is contained between  $-\frac{f_s}{2}$  and  $\frac{f_s}{2}$ .
  - ii)  $f_r = f_o$  in the above window
  - iii) when  $f_o$  exceed  $\frac{f_s}{2}$ , it "jumps" (or "goes through the tunnel symbolized by the dotted line") back to restart at  $-\frac{f_s}{2}$ .
  - iv) as a function of  $f_o$ ,  $f_r$  is periodic with period  $f_s$ .
- $\Rightarrow$  Therefore, instead of considering  $f_o$ , we may simply consider  $f_t = f_o \bmod f_s$

This is implemented in MATLAB by  $\text{mod}(f_o, f_s)$

Alternatively, one can use

$$f_t = f_o - f_s \lfloor \frac{f_o}{f_s} \rfloor$$

This leads to method ② below

Method 2: Use the "tunneling effect" discussed in class (wherein we observe the location of the impulse(s) shown by our plotspect function).

I) Find  $f_t = f_o \bmod f_s$  ← This gives  $f_t \in [0, f_s)$

or, equivalently,  $f_t = f_o - f_s \lfloor \frac{f_o}{f_s} \rfloor$

Think of this as representing the number of "rounds" you start from 0, go through the tunnel, and back to 0.

$$\text{II) } f_r = \begin{cases} f_t, & \text{if } f_t \leq \frac{f_s}{2}, \\ f_t - f_s, & \text{if } f_t > \frac{f_s}{2}. \end{cases}$$

In part (c),  $f_o = 11,111 \Rightarrow f_t = f_o \bmod f_s = 11 > 6$   
 $\Rightarrow f_r = f_t - f_s = 11 - 12 = -1.$

In part (d),  $f_o = -1,111 \Rightarrow f_t = f_o \bmod f_s = 5 \leq 6$   
 $\Rightarrow f_r = f_t = 5.$

Finding the "perceived" frequency of  $g(t) = \cos(2\pi f_c t)$  when the sampling rate is  $f_s$ .

Here, we write  $f_c$  instead of  $f_0$  because there are two terms with two freq.:  $f_0 = f_c$   
and  $f_0 = -f_c$

Method 1: From the Euler's formula:  $g(t) = \cos(2\pi f_c t) = \frac{1}{2} e^{j2\pi f_c t} + \frac{1}{2} e^{j2\pi (-f_c) t}$ .  
After sampling, we can apply what we know about the "perceived" frequency of complex-exponential signal to get the "perceived" freq. of each term inside  $g(t)$ .

In part (a),  $f_c = 1,111,111 \Rightarrow f_t = f_c \bmod f_s = 7 > 6 \leftarrow f_s/2$   
 $\Rightarrow f_r = f_t - f_s = 7 - 12 = -5$   
 $-f_c = -1,111,111 \Rightarrow f_t = (-f_c) \bmod f_s = 5 \leq 6$   
 $\Rightarrow f_r = f_t = 5$

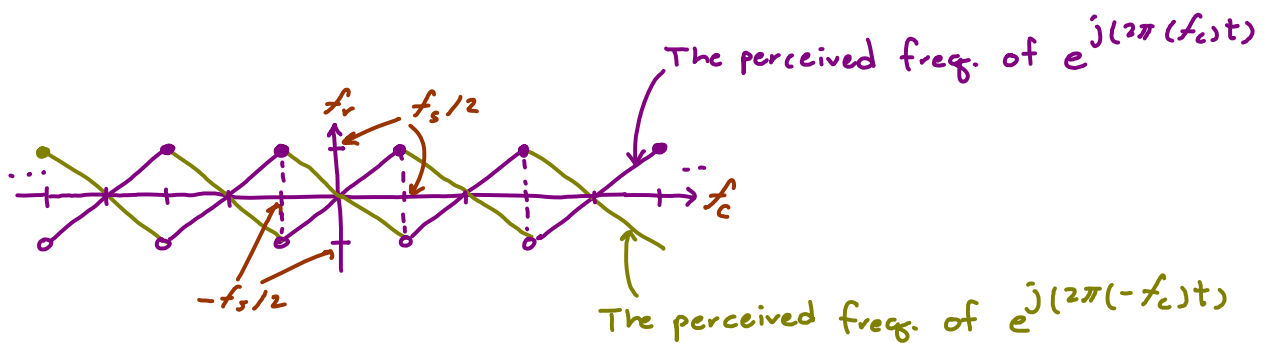
Therefore,  $g_r(t) = \frac{1}{2} e^{j(2\pi(-5)t)} + \frac{1}{2} e^{j(2\pi(5)t)} = \cos(2\pi(5)t)$ .  
 The "perceived" frequency is 5 Hz.

In part (b),  $f_c = 111,111 \Rightarrow f_t = 3 \leq 6 \Rightarrow f_r = 3$

$-f_c = -111,111 \Rightarrow f_t = 9 > 6 \Rightarrow f_r = 9 - 12 = -3$

Therefore,  $g_r(t) = \frac{1}{2} e^{j(2\pi(3)t)} + \frac{1}{2} e^{j(2\pi(-3)t)} = \cos(2\pi(3)t)$   
 The "perceived" freq. is 3 Hz.

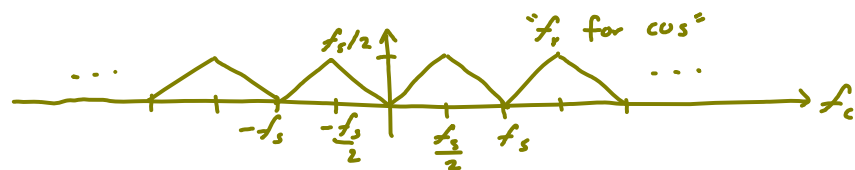
Method 2: When we consider the "perceived freq." of  $e^{j2\pi f_c t}$  which we plotted earlier with " "  $e^{j2\pi(-f_c)t}$



Observe that at every  $f_c$ ,  
for  $\cos(2\pi f_c t)$

its two complex-expo components always gives a pair of perceived freq., one positive and one negative (except when  $f_c$  is a multiple of  $f_s/2$ )

so, the reconstructed signal  $g_r(t)$  will still be a cosine whose freq. can simply be "read" from the upper part of the plot above:



(Because  $\cos(-x) = \cos(x)$ , we only answer one freq. for the cosine.)

Observe the "folding effect":

- " $f_r$  for  $\cos$ " is contained between 0 and  $\frac{f_s}{2}$
- " " =  $f_c$  in the above window
- when  $f_c$  exceeds  $\frac{f_s}{2}$ , it folds back towards 0.  
when  $f_c$  reaches 0, it folds back towards  $\frac{f_s}{2}$ .
- as a function of  $f_c$ ,

" $f_r$  for  $\cos$ " is periodic with period  $f_s$ .

Therefore, we can find " $f_r$  for  $\cos$ " by

$$\text{I) find } f_t = f_c \bmod f_s = f_c - f_s \left\lfloor \frac{f_c}{f_s} \right\rfloor.$$

$$\text{II) } f_r = \begin{cases} f_t, & \text{if } f_t < \frac{f_s}{2}, \\ f_s - f_t, & \text{if } f_t > \frac{f_s}{2}. \end{cases}$$

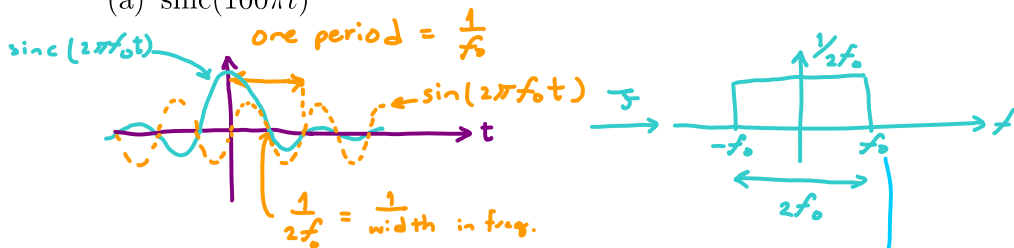
In part (a),  $f_c = 1,111,111 \Rightarrow f_t = 7 > 6 \Rightarrow f_r = 12 - 7 = 5 \text{ Hz}$ .

In part (b),  $f_c = 111,111 \Rightarrow f_t = 3 < 6 \Rightarrow f_r = 3 \text{ Hz}$ .

Remark: In lecture, we defined the “perceived” frequency to be the frequency that plotspect “sees” the signal. Now that we learned about reconstruction, we can define the “perceived” frequency to be the frequency of the signal reconstructed by the “reconstruction equation”. If the reconstructed signal is  $\cos(2\pi\tilde{f}_0t)$  where  $\tilde{f}_0 \geq 0$ , the perceived frequency is  $\tilde{f}_0$ . If the reconstructed signal is  $e^{j2\pi\tilde{f}_0t}$ , the perceived frequency is  $\tilde{f}_0$ . Note that for complex-exponential signal, the perceived frequency can be negative.

**Problem 2.** Determine the Nyquist sampling rate and the Nyquist sampling interval for the following signals: *First, we need to look at the freq. domain representation of the given signal to find the maximum non-zero frequency B.*

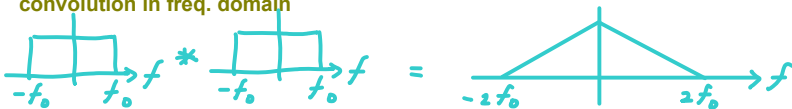
(a)  $\text{sinc}(100\pi t)$   
 $2\pi f_0 t \Rightarrow f_0 = 50$



Conclusion:  $\text{sinc}(2\pi f_0 t)$  is bandlimited to  $B = f_0$

(b)  $\text{sinc}^2(100\pi t)$   
 multiplication in time domain

convolution in freq. domain

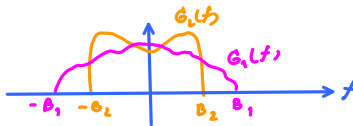


Conclusion:  $\text{sinc}^2(2\pi f_0 t)$  is bandlimited to  $\pm 2f_0$

(c)  $\text{sinc}(100\pi t) + \text{sinc}(50\pi t)$

Observe that for signals  $g_1(t)$  bandlimited to  $B_1$  and  $g_2(t)$  " "  $B_2$ ,

their linear combination  $c_1 g_1(t) + c_2 g_2(t)$  is bandlimited to  $\max\{B_1, B_2\}$ .



so, for  $\text{sinc}(100\pi t) + \text{sinc}(50\pi t)$ ,  $f_{max} = \max\{50, 25\} = 50 \text{ Hz}$

(d)  $\text{sinc}(100\pi t) + 3\text{sinc}^2(60\pi t)$

Use the observation from parts (b) and (c).

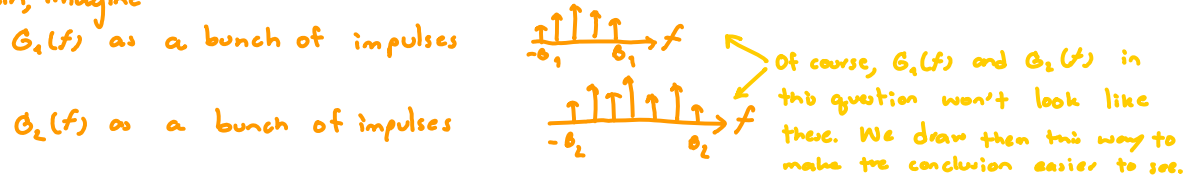
For  $\text{sinc}(100\pi t) + 3\text{sinc}^2(60\pi t)$ ,  $f_{max} = \max\{50, 2 \times 30\} = 60 \text{ Hz}$ .

	$f_{max}$	$R_{\text{Nyquist}}$ [Sa/s]	$T_{\text{Nyquist}}$ [Sec]
(a)	50	100	0.01
(b)	100	200	0.005
(c)	50	100	0.01
(d)	60	120	1/120
(e)	75	150	1/150

Note that for signals  $g_1(t)$  bandlimited to  $B_1$  and  $g_2(t)$  " "  $B_2$ ,

their product  $g_1(t)g_2(t)$  is bandlimited to  $B_1+B_2$ .

To "see" this (without actually doing the "flip-shift-integrate" for convolution in the freq. domain, imagine



Because we have a multiplication in the time domain, we have a convolution in the frequency domain.

The convolution with an impulse is easy:

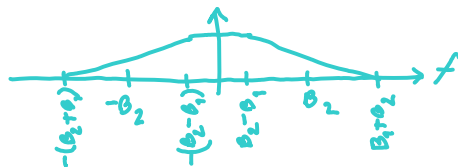
$$G_1(f) * \delta(f-f_0) = G_1(f-f_0).$$

So, we simply have replicas of  $G_1(f)$  at all the impulses' locations of  $G_2(f)$ . Hence, the highest freq. component is at  $B_1+B_2$  and the lowest freq. component is at  $-B_1-B_2$ .

Alternatively, one can look at the convolution of two rectangular functions:



We saw similar convolution as a video earlier in the semester. The result is



(e)  $\text{sinc}(50\pi t) \text{sinc}(100\pi t)$

For  $\text{sinc}(50\pi t) \text{sinc}(100\pi t)$ , the observation on the previous page gives

$$f_{\max} = 25 + 50 = 75 \text{ Hz.}$$

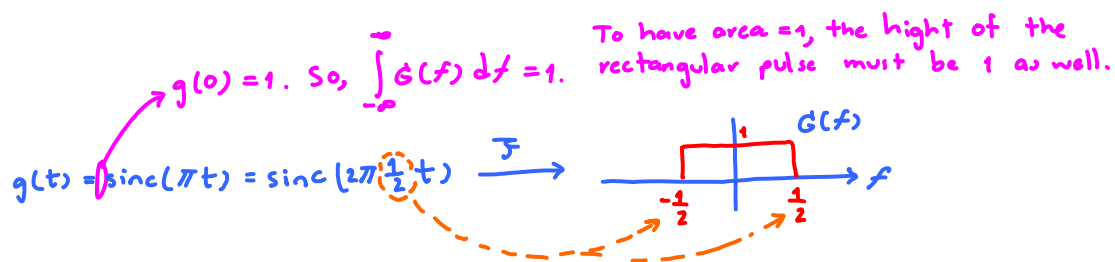
Remark: Recall that in our class,  $\text{sinc}(x) = \frac{\sin(x)}{x}$ .

**Problem 3.** Consider a signal  $g(t) = \text{sinc}(\pi t)$ .



(a) Sketch its Fourier transform  $G(f)$ .

note that, in MATLAB, this function is implemented by `sinc(t)` because the built-in MATLAB `sinc` function has already included the  $\pi$ .



(b) Find the Nyquist sampling rate.

$$2 \times f_{\max} = 2 \times \frac{1}{2} = 1 \text{ sample/sec.}$$

(c) The sequence of sampled values  $g[n]$  is constructed from  $g(t)$  by

$$g[n] = g(t)|_{t=nT_s}.$$

Recall that the instantaneous sampled signal  $g_\delta(t)$  is defined by

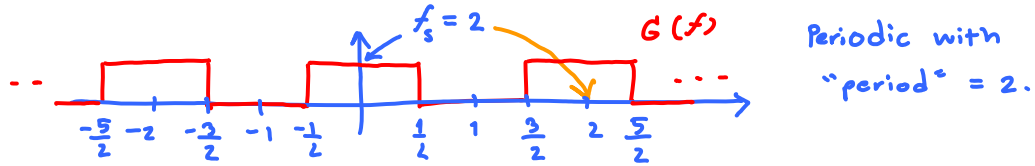
$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n] \delta(t - nT_s)$$

where  $T_s$  is the sampling interval.

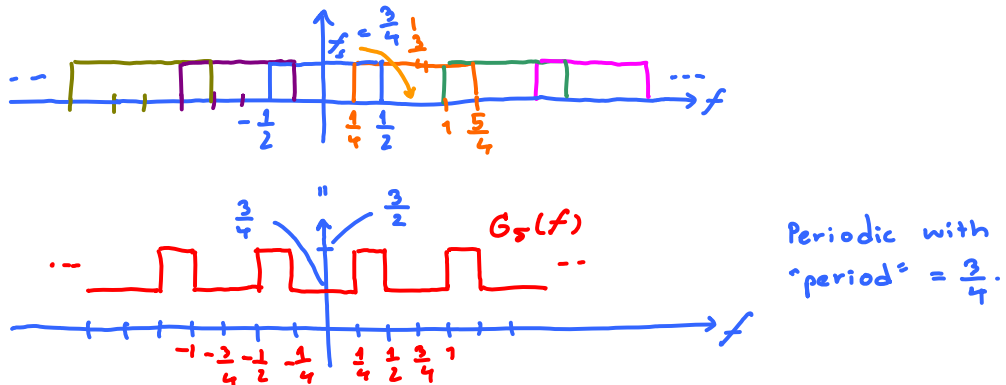


$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - n f_s) \quad \text{where } f_s = \frac{1}{T_s}.$$

- (i) Let  $T_s = 0.5$ . Sketch the Fourier transform  $G_\delta(f)$  of  $g_\delta(t)$ .



- (ii) Let  $T_s = 4/3$ . Sketch the Fourier transform  $G_\delta(f)$  of  $g_\delta(t)$ .



- (d) Recall the reconstruction equation:

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g[n] \operatorname{sinc}(\pi f_s(t - nT_s)). \quad (7.1)$$

Note that we write  $\hat{g}(t)$  instead of  $g(t)$  to accommodate the case that the sampling rate is too low; in which case, the reconstructed signal is not the same as  $g(t)$ .

With  $T_s = 1$ ,  $g[n] = g(nT_s) = g(n \times 1) = g(n) = \operatorname{sinc}(\pi n)$ .

- (i) Find  $g[n]$  for  $n = \dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots$

From the plot of  $\operatorname{sinc}(\pi t)$  drawn earlier, we have

$$g[n] = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- (ii) Use the reconstruction equation to find  $\hat{g}(t)$ .

Because  $g[n] = 0$  when  $n \neq 0$ ,

$$\hat{g}(t) = g[0] \operatorname{sinc}(\pi t) = 1 \times \operatorname{sinc}(\pi t) = \operatorname{sinc}(\pi t) = g(t)$$

Note that with  $T_s = 1$ , we have  $f_s = 1$  which is the same as the Nyquist sampling rate. Therefore, we are at the borderline of the successful reconstruction.

## Extra Questions

Here are some optional questions for those who want more practice.

**Problem 4.** We can use MATLAB to study the effect of sampling in both time and frequency domain. Consider a bipolar square pulse periodic signal:

$$g(t) = \text{sgn}(\cos(2\pi f_0 t)) = \begin{cases} 1, & \cos(2\pi f_0 t) \geq 0, \\ -1, & \cos(2\pi f_0 t) < 0. \end{cases}$$

- (a) When  $f_0 = 10$  [Hz], plot  $g(t)$  from  $t = 0$  [s] to  $t = 0.5$  [s].
- (b) Note that the bipolar square pulse periodic signal is almost the same as our ON-OFF signal studied in lecture. There,

$$g_{\text{ON-OFF}}(t) = 1[\cos(2\pi f_0 t) \geq 0] = \begin{cases} 1, & \cos(2\pi f_0 t) \geq 0, \\ 0, & \cos(2\pi f_0 t) < 0. \end{cases}$$

The ON-OFF signal studied in lecture alternates between “0 and 1”. Here,  $g(t)$  alternates between “-1 and 1”. Find the constant  $a$  and  $b$  such that

$$g(t) = a \times g_{\text{ON-OFF}}(t) + b.$$

- (c) The (magnitude) spectrum of a periodic square wave can be found using:

```
% specsquare.m plot the spectrum of a square wave
close all
f0=10; % "(fundamental) frequency" of the square wave
EndTime=2; % Will consider from time = 0 to EndTime
Ts=1/1000; % sampling interval
t=0:Ts:(EndTime-Ts); % create a time vector
g=sign(cos(2*pi*f0*t)); % square wave = sign of cos wave
plotspect(g,t) % call plotspec to draw spectrum
```

The output of `specsquare.m` is shown in Figure 7.1. The top plot shows the first 2 seconds of  $g(t)$  with fundamental frequency  $f_0 = 10$  cycles per second. The bottom plot shows a series of spikes that define its frequency content.

In lecture, we showed that

$$g_{\text{ON-OFF}}(t) = 1[\cos(2\pi f_0 t) \geq 0] \\ = \frac{1}{2} + \frac{2}{\pi} \left( \cos 2\pi f_0 t - \frac{1}{3} \cos 2\pi 3f_0 t + \frac{1}{5} \cos 2\pi 5f_0 t - \frac{1}{7} \cos 2\pi 7f_0 t + \dots \right).$$

- (i) Use the relationship derived in part (b) to find the Fourier series expansion of  $g(t)$ .

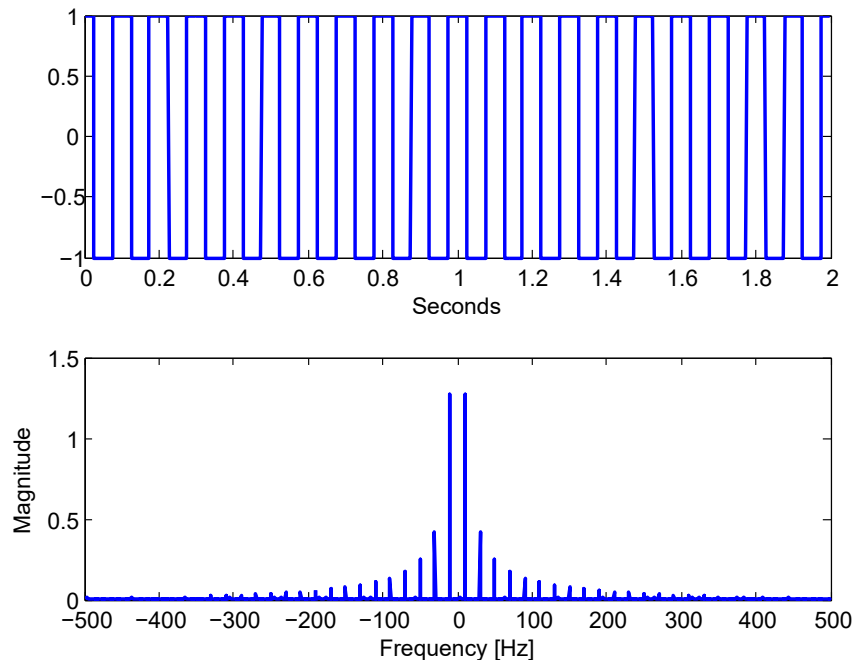


Figure 7.1: Plots from `specsquare.m`

(ii) Use the Fourier series expansion in the previous part to double-check the lower plot in Figure 7.1.

- i. Do the spikes show up at the expected locations?
- ii. Note that, in the script, we consider the time between 0 to 2 [s]. Therefore, actually, we are not looking at the signal  $g(t)$  from  $-\infty$  to  $\infty$ . This time-limited view means that, in the frequency domain, we won't see the impulses but rather sinc pulses at the expected locations. (This is the same as seeing two sinc pulses instead of two impulses when looking at the Fourier transform of the cosine pulse.)

In Figure 7.1, the height of the highest spikes in the center is about 1.2 or 1.3. Calculate (derive analytically) the exact height of these spikes.

(d) In this part, we increase the value of  $f_0$  in `specsquare.m` to 20, 40, 100, and 300 Hz while keeping the sampling interval at  $T_s = 1/1000$ . The magnitude spectrum plots (the lower subplots from `specsquare.m`) for these cases are shown in Figure 7.2.

Use the same analysis as in part c.(ii).i and the “folding technique” to explain (the aliasing effect in) each of the provided plots

**Problem 5.** Again, consider a signal  $g(t) = \text{sinc}(\pi t)$ . Let's test the reconstruction equation (7.1) by using MATLAB to plot  $\hat{g}(t)$ . Note that the sum in the reconstruction equation extends from  $-\infty$  to  $+\infty$ . In MATLAB, we can not add that many terms. So, we need to stop at some  $n$ . In this problem, use  $T_s = 0.5$ .

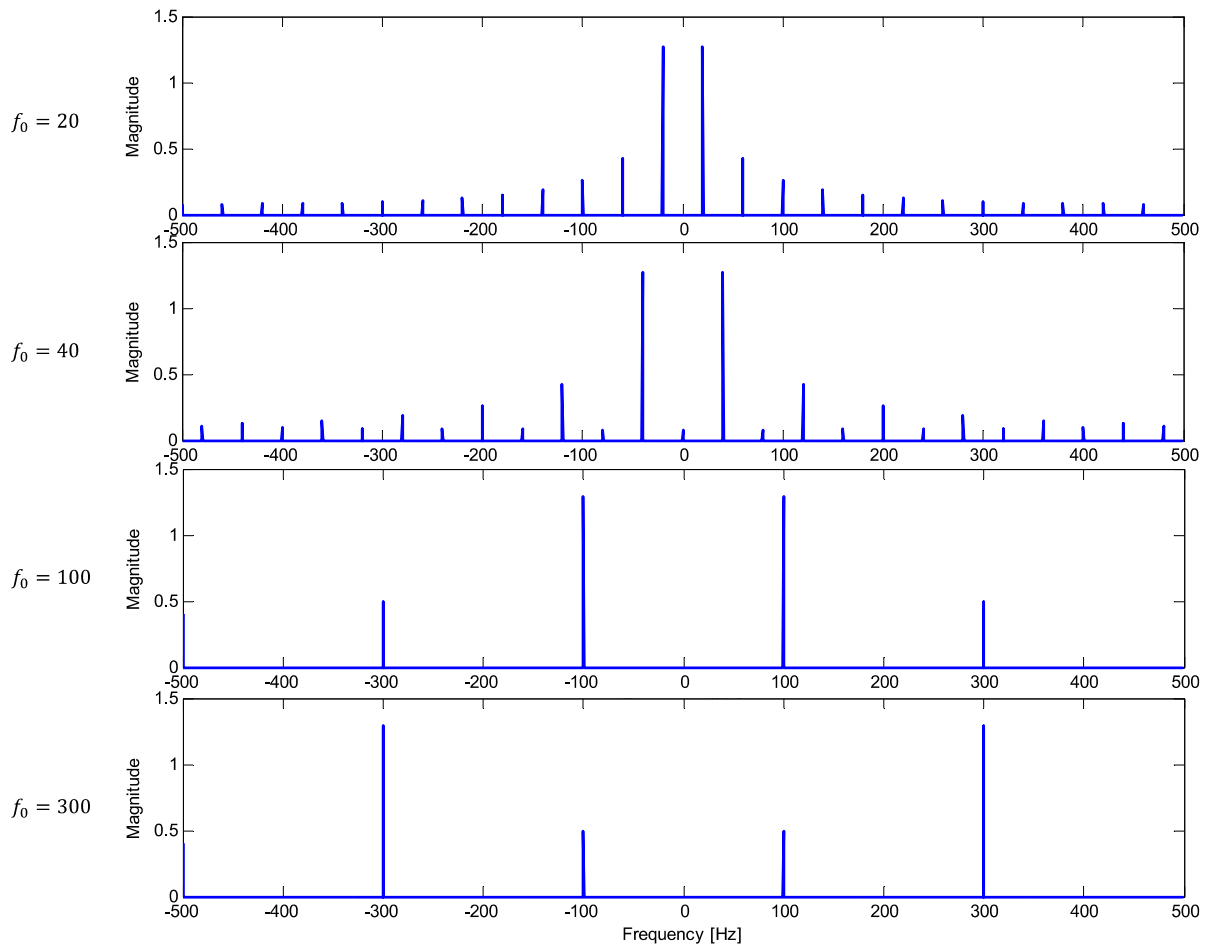


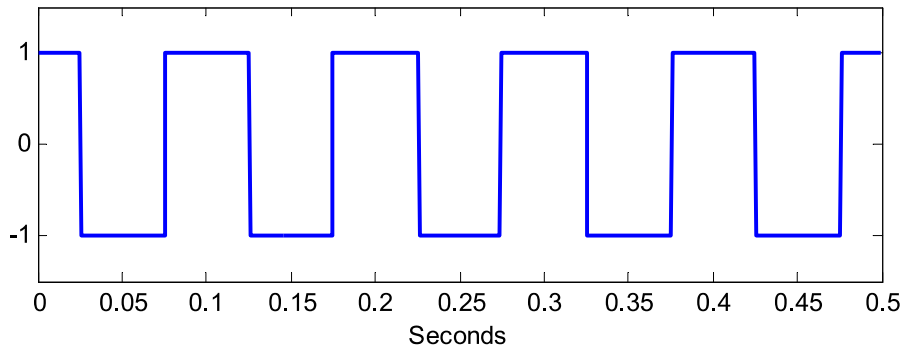
Figure 7.2:

- Use MATLAB to plot  $\hat{g}(t)$  when only the  $n = 0$  term is included.
- Use MATLAB to plot  $\hat{g}(t)$  and all of its sinc components. Include only  $n = -1, 0, 1$ .
- Use MATLAB to plot  $\hat{g}(t)$  and all of its sinc components. Include only  $n = -5, -4, \dots, -1, 0, 1, \dots, 4, 5$ .
- Use MATLAB to plot  $\hat{g}(t)$  and all of its sinc components. Include only  $n = -10, -9, \dots, -1, 0, 1, \dots, 9, 10$ .

In all these plots, consider  $t$  from -4 to 4. Also include the plot of  $\text{sinc}(\pi t)$  for comparison.

## Problem 4: Aliasing and periodic square wave

(a)



(b)  $\text{sgn}(\cos\omega_0 t) = 2 \times 1[\cos\omega_0 t \geq 0] - 1$ . So, **a = 2** and **b = -1**.

(c)

(c.i) From

$$1[\cos\omega_0 t \geq 0] = \frac{1}{2} + \frac{2}{\pi} \left( \cos\omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t - \frac{1}{7} \cos 7\omega_0 t + \dots \right),$$

and

$$g(t) = \text{sgn}(\cos\omega_0 t) = 2 \times 1[\cos\omega_0 t \geq 0] - 1$$

we have

$$g(t) = \text{sgn}(\cos\omega_0 t) = \frac{4}{\pi} \left( \cos\omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t - \frac{1}{7} \cos 7\omega_0 t + \dots \right).$$

where  $\omega_0 = 2\pi f_0$ .

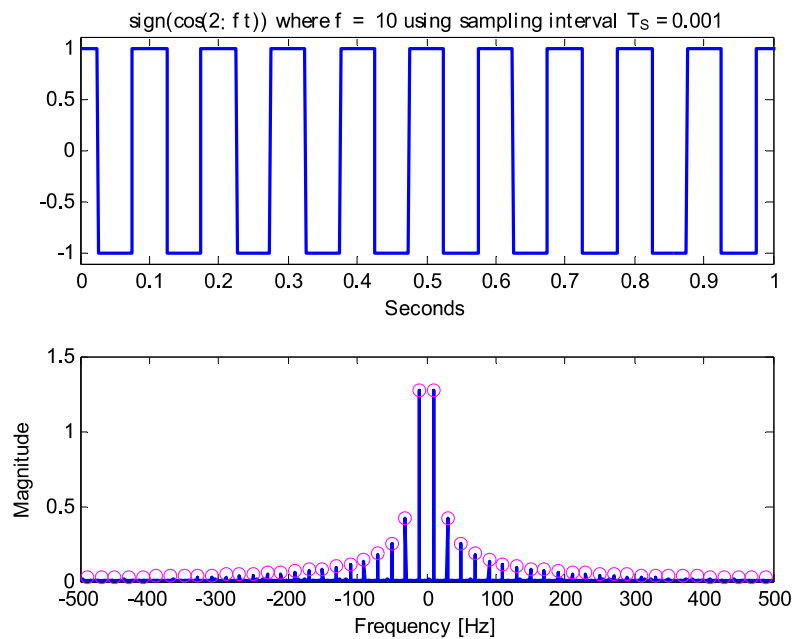
(c.ii)

(c.ii.i) Theoretically,  $G(f)$  should have spikes (impulses) at all the odd-integer multiples of  $\pm f_0$  Hz. The center spikes (at  $\pm f_0$ ) should be the largest among them. Here,  $f_0 = 10$ . Therefore, the largest spike occurs at  $\pm 10$  Hz, followed by smaller spikes at all the odd-integer multiples (i.e., at  $\pm 30, \pm 50, \pm 70$ , etc).

(c.ii.ii) The sinc function is simply the Fourier transform of the rectangular windows. Because the area of the rectangular window is  $1 \times 2 = 2$ , its Fourier transform (which is a sinc function) has its peak value of 2. This is further scaled by a factor

of  $\frac{1}{2}$  from the cosine. Therefore, each “impulse” (“sinc”) that we see should have its height being the same as the coefficient of corresponding cosine. For example, at  $\pm f_0$ , the coefficient of the cosine is  $\frac{4}{\pi}$ . Therefore, we expect the height of the “impulse” at  $\pm f_0$  to be  $\frac{4}{\pi} \approx 1.2732$ .

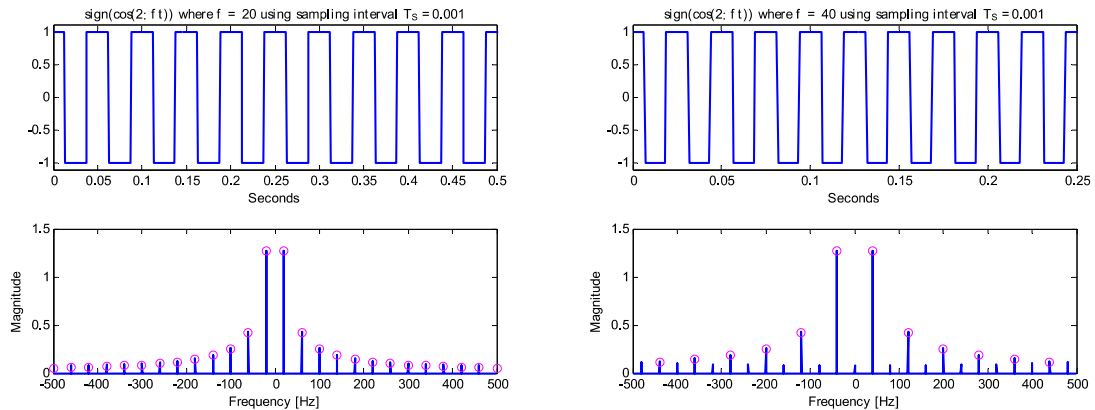
The theoretical values for the height of other impulses is shown by the pink circles in the plot. We see that our predicted values match the plot quite well.



(d) Here, we also shows the time-domain plots as well. In the time domain, the switching between the values -1 and 1 should be faster as we increase  $f_0$ . All the plots here are adjusted so that they show 10 periods of the “original signal” in the time domain. (This is done so that the distorted shape (if any) of the waveform in the time domain is visible.)

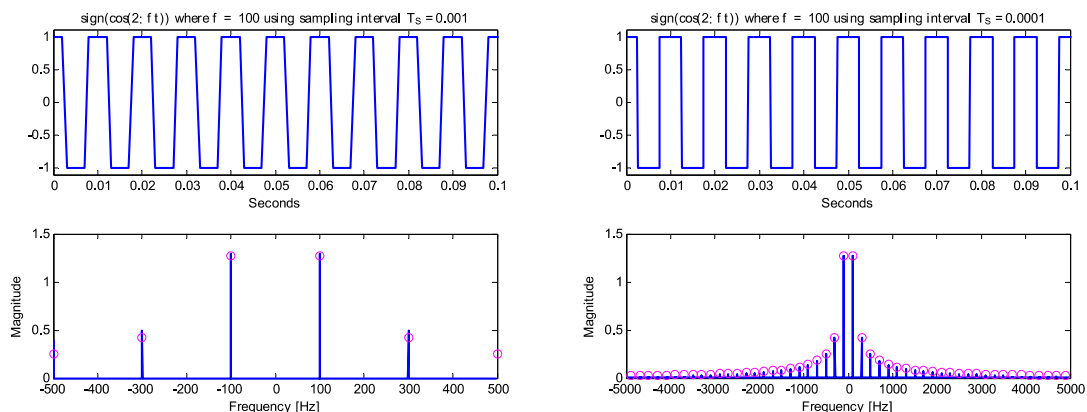
(d.i) From the plots, as we increase  $f_0$  from 10 to **20 Hz**, the **locations of spikes** changes from all the odd-integer multiples of 10 Hz to all the **odd-integer multiples of 20 Hz**. In particular, we see the spikes at  $\pm 20, \pm 60, \pm 100, \pm 140, \pm 180, \pm 220, \pm 260, \pm 300, \pm 340, \pm 380, \pm 420, \pm 460$ . Note that `plotspect` (by the way that it is coded) only plots from  $[-f_s/2, f_s/2)$ . So, we see a spike at -500 but not 500. Of course, the Fourier transform of the sampled waveform is periodic and hence when we replicate the spectrum every  $f_s$ , we will have a spike at 500. Note that, in theory, we should also see spikes at  $\pm 540, \pm 580, \pm 620, \pm 660$ , and so on. However, because the sampling rate is 1000 [Sa/s], these high

frequency spikes will suffer from aliasing and “fold back”<sup>1</sup> into our viewing window  $[-f_s/2, f_s/2)$ . However, they fall back to the frequencies that already have spikes (for example,  $\pm 540$  will fold back to  $\pm 460$ , and  $\pm 580$  will fold back to  $\pm 420$ ) and therefore the aliasing effect is not easily noticeable in the frequency domain.



(d.ii) When  $f_0 = 40$  Hz, we start to see the aliasing effect in the frequency domain. Instead of seeing spikes only at  $\pm 40, \pm 120, \pm 200, \pm 280, \pm 360, \pm 440$ , the spikes at higher frequencies (such as  $\pm 520, \pm 600$ , and so on) fold back to lower frequencies (such as  $\pm 480, \pm 400$ , and so on). The plot in the time domain still looks quite OK with small visible distortion.

(d.iii) At high fundamental frequency  $f_0 = 100$  Hz, we see stronger effect of aliasing. In the time domain, the waveform does not look quite “rectangular”.



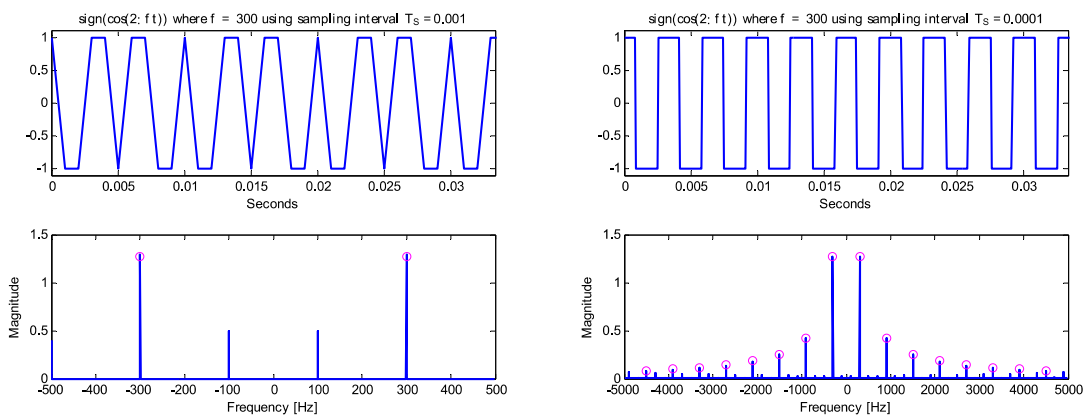
In the frequency domain, we only see the spikes at  $\pm 100, \pm 300$ , and  $500$ . These are at the correct locations. However, there are too few of them to reconstruct a square waveform. The rest of the spikes are beyond our viewing window. We can't see them

<sup>1</sup> Because the squarewave is real and even, the Fourier transform is also real and even. Therefore, the “folding effect” is “equivalent” to the “tunneling effect”.

directly because they **fold back to the frequencies that are already occupied** by the lower frequencies. Note also that the predicted height (**pink circles**) at  $\pm 300$  Hz is quite different from the `plot_spect` value. This is because the content from the folded-back higher-frequencies is being combined into the spikes.

Our problem can be mitigated by reducing the sampling interval to  $T_s = 1/1e4$  instead of  $T_s = 1/1e3$  as shown by the plot on the right above.

(d.iv) Finally, at the highest frequency  $f_0 = 300$  Hz, if we still use  $T = 1/1e3$ , **the waveform will be heavily distorted in the time domain. This is shown in the left plot below. We have large spikes at  $\pm 300$  as expected. However, the next pair which should occur at  $\pm 900$  is out of the viewing window and therefore  **folds back to  $\pm 100$ .****



Again, the aliasing effect can be mitigated by reducing the sampling time to  $T = 1/1e4$  instead of  $T = 1/1e3$ . Now, more spikes show up at their expected places. Note that we can still see a lot of small spikes scattered across the frequency domain. These are again the spikes from higher frequency which fold back to our viewing window.

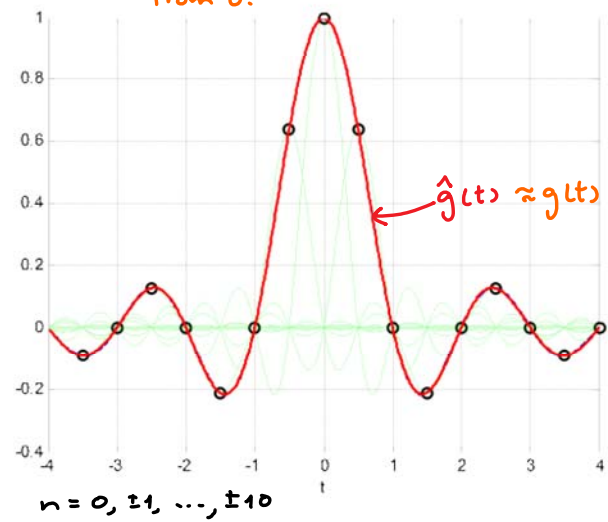
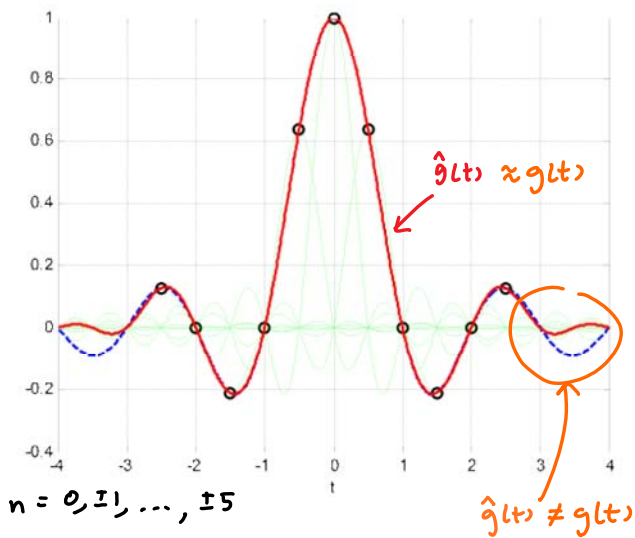
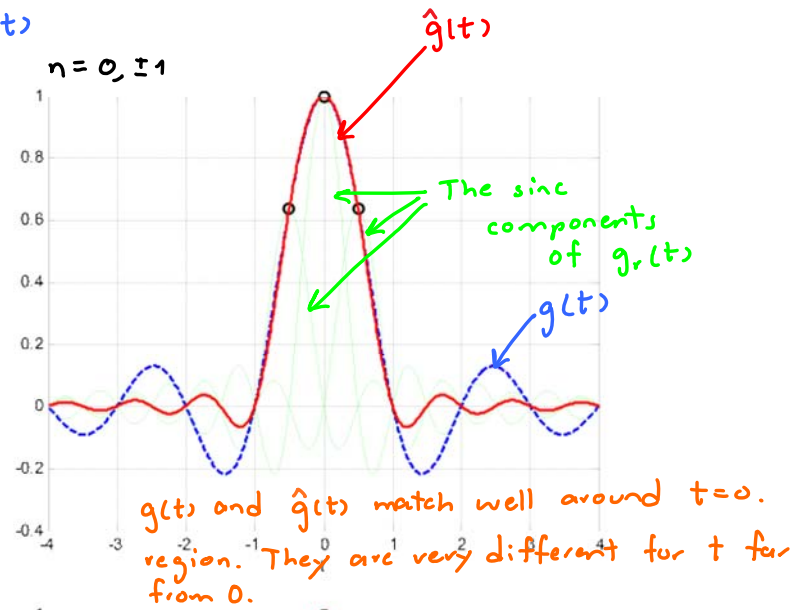
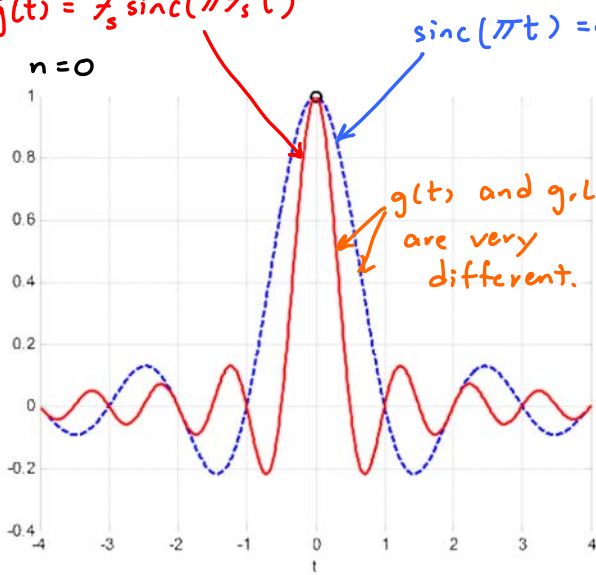


# Q5 Sinc Reconstruction of Sinc: MATLAB Simulation

Wednesday, November 16, 2016 8:34 PM

Reminder: MATLAB's sinc function is  $\text{sinc}(x) = \sin(\pi x) / \pi x$ , which is different from  $\text{sinc}(x) = \sin(x) / x$  that we defined for our class. Therefore, when we use MATLAB to plot  $\text{sinc}(\pi t)$ , we do not put the " $\pi$ " in the formula. MATLAB will automatically insert the " $\pi$ " for us.

$$\hat{g}(t) = \sum_s \text{sinc}(\pi f_s t)$$



Observation: As we increase the number of terms in the summation,  $g(t)$  is better approximated by  $\hat{g}(t)$ .