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ECS 332: Principles of Communications **2017/1**

HW 5 — Due: Nov 3, 4 PM **Solution**

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Instructions

(a) This assignment has 12 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider a "square" wave (a train of rectangular pulses) shown in Figure 5.1. Its values periodically alternates between two values A and 0 with period T_0. At $t = 0$, its value is A.

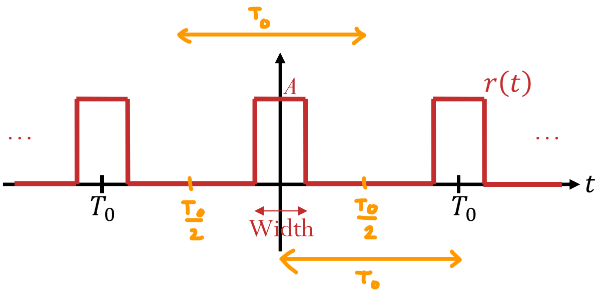


Figure 5.1: A train of rectangular pulses

First, recall that

duty cycle $= \frac{\text{width}}{T_0}$

Some values of its Fourier series coefficients are provided in the table below:

k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
c_k	$-\frac{\sqrt{2}}{7\pi}$	$-\frac{1}{3\pi}$	$-\frac{\sqrt{2}}{5\pi}$	0	$\frac{\sqrt{2}}{3\pi}$	$\frac{1}{\pi}$	$\frac{\sqrt{2}}{\pi}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{\pi}$	$\frac{1}{\pi}$	$\frac{\sqrt{2}}{3\pi}$	0	$-\frac{\sqrt{2}}{5\pi}$	$-\frac{1}{3\pi}$	$-\frac{\sqrt{2}}{7\pi}$

(a) Find its duty cycle.

In class, we've seen that when the duty cycle is $\frac{1}{n}$, the n^{th} harmonic (along with its multiples) is suppressed.

Here, $c_4 = 0$. So, we conclude that the duty cycle is $\frac{1}{4} = 25\%$.

(b) Find the value of A. (Hint: Use c_0.)

Recall that $c_k = \frac{1}{T_0}\int_{T_0} r(t)\, e^{-j2\pi k f_0 t}\, dt$.

Therefore, $c_0 = \frac{1}{T_0}\int_{T_0} r(t)\, dt = \underbrace{\langle r(t)\rangle}_{\text{time average}}$.

From the picture, $\langle r(t)\rangle = \frac{\text{width} \times A}{T_0} = (\text{duty cycle}) \times A$.

Therefore, $A = \frac{\langle r(t)\rangle}{\text{duty cycle}}$

We are given that $c_0 = \frac{1}{2}$ and we found, in part (a), that duty cycle $= \frac{1}{4}$.

Therefore, $A = \frac{1/2}{1/4} = 2$.

Problem 2. You are asked to design a DSB-SC modulator to generate a modulated signal $km(t)\cos(2\pi f_c t)$, where $m(t)$ is a signal band-limited to B Hz. Figure 5.2 shows a DSB-SC modulator available in the stockroom. Note that, as usual, $\omega_c = 2\pi f_c$. The carrier generator available generates not $\cos(2\pi f_c t)$, but $\cos^3(2\pi f_c t)$. Explain whether you would be able to generate the desired signal using only this equipment. You may use any kind of filter you like. [Lathi and Ding, 2009, Q4.2-3]

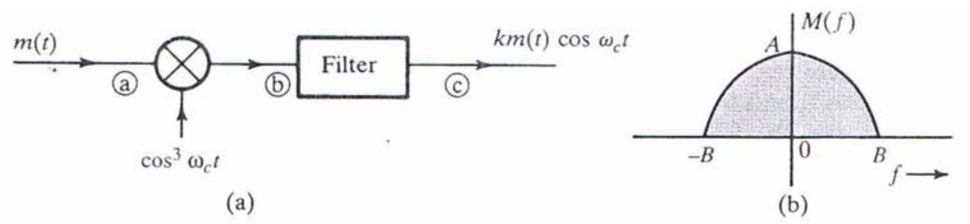


Figure 5.2: Problem 2 Powered Cosine Modulator

(a) We know that a real-valued signal $r(t)$ that is even and periodic with period T_0 can be expanded using Fourier series into

$$r(t) = c_0 + a_1 \cos(2\pi f_0 t) + a_2 \cos(2\pi(2f_0)t) + a_3 \cos(2\pi(3f_0)t) + \cdots \quad (5.1)$$

where $f_0 = \frac{1}{T_0}$. Consider the signal $r(t) = \cos^3(2\pi f_c t)$.

(i) Is it periodic?

Yes. The function $\cos(2\pi f_c t)$ is periodic. Therefore, $r(t)$ is a function of another periodic function and hence it is periodic.

(ii) Is it even?

$\cos(-x) = \cos(x)$

Yes. $r(-t) = \cos^3(2\pi f_c(-t)) = (\cos(2\pi f_c t))^3 = \cos^3(2\pi f_c t) = r(t)$

(iii) Expand $r(t) = \cos^3(2\pi f_c t)$ into a linear combination of $\cos(2\pi(nf_c)t)$ as in (5.1) above.

First, we use the product-to-sum formula

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

to expand $\cos^3 x$ into sum of weighted $\cos(kx)$.

$$\cos^2 x = \cos x \cos x = \frac{1}{2}\left(\cos(2x) + \cos(0)\right) = \frac{1}{2}(\cos 2x + 1)$$

$$\cos^3 x = \cos x \cos^2 x = \cos x\left(\frac{1}{2}(\cos 2x + 1)\right) = \frac{1}{2}\Big(\underbrace{\cos x \cos 2x}_{=\frac{1}{2}(\cos(3x)+\cos x)} + \cos x\Big) = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

Alternatively,

$$(A+B)^3 = \binom{3}{0}A^0B^3 + \binom{3}{1}A^1B^2 + \binom{3}{2}A^2B^1 + \binom{3}{3}A^3B^0$$

$$\cos^3 x = \left(\frac{1}{2}(e^{jx} + e^{-jx})\right)^3 = \frac{1}{8}\left(e^{-3jx} + 3e^{-jx} + 3e^{jx} + e^{3jx}\right) = \frac{1}{8}\left(2\cos(3x) + 6\cos(x)\right)$$

Plugging in $x = \omega_c t = 2\pi f_c t$, we get $\cos^3 \omega_c t = \frac{3}{4}\cos(\omega_c t) + \frac{1}{4}\cos(3\omega_c t)$.

$$\cos^3(2\pi f_c t) = \frac{3}{4}\cos(2\pi f_c t) + \frac{1}{4}\cos(2\pi(3f_c)t)$$

Compare this with eqn. (5.1). We have $f_0 = f_c$, $c_0 = 0$, $a_1 = \frac{3}{4}$, $a_2 = 0$, $a_3 = \frac{1}{4}$, $a_n = 0$ for $n \geq 4$

(b) What kind of filter is required in Figure 5.2?

See 4.56 in the lecture notes.

At point ⓒ, we want $k\,m(t)\cos\omega_c t$

At point ⓑ, we have $m(t)\cos^3\omega_c t = \underbrace{\frac{1}{4}m(t)\cos(3\omega_c t)}_{\text{don't want this part}} + \underbrace{\frac{3}{4}m(t)\cos(\omega_c t)}_{\text{want this part}}$.

Any bandpass filter centered at $\pm f_c$ will work.

In addition, the passband of this filter must be larger than 2B.

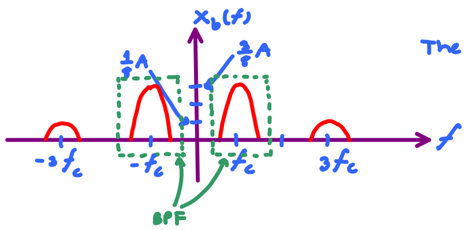
Note that if the gain of the BPF is 1, then $k = g a_1 = 1 \times \frac{3}{4} = \frac{3}{4}$.

(c) Determine the signal spectra at points (b) and (c) in Figure 5.2, and indicate the frequency bands occupied by these spectra.

(b.1) Let $x_b(t)$ be the signal at point (b).

Then $x_b(t) = m(t)\cos^3 \omega_c t = \frac{1}{4} m(t)\cos(3\omega_c t) + \frac{3}{4} m(t)\cos(\omega_c t)$

$\xrightarrow{\mathcal{F}} \frac{1}{8} M(f-3f_c) + \frac{1}{8} M(f+3f_c) + \frac{3}{8} M(f-f_c) + \frac{3}{8} M(f+f_c)$ where $f_c = \omega_c/2\pi$.



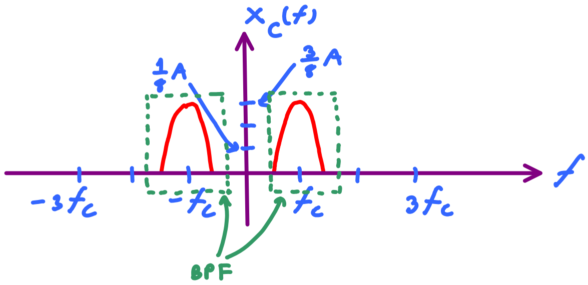
The frequency bands occupied are $[-3f_c-B, -3f_c+B]$, $[-f_c-B, -f_c+B]$, $[f_c-B, f_c+B]$, and $[3f_c-B, 3f_c+B]$.

(b.2) Let $x_c(t)$ be the signal at point (c).

We will assume that the gain of the BPF is 1. (In general, if gain $= g$, then $k = \frac{3}{4} g$)

In which case, $x_c(t) = \frac{3}{4} m(t) \cos \omega_c t$

and $X_c(f) = \frac{3}{8} M(f-f_c) + \frac{3}{8} M(f+f_c)$



The frequency bands occupied are

$[-f_c-B, -f_c+B]$ and $[f_c-B, f_c+B]$

(d) What is the minimum usable value of f_c?

To avoid overlapping of spectra at point (b),

we must have $f_c - B > 0$, and $f_c + B < 3f_c - B$.

Both conditions require $f_c > B$. Hence, the minimum usable value of f_c is B.

(e) Would this scheme work if the carrier generator output were $\cos^2(2\pi f_c t)$? Explain.

Recall (from part (a)) that $\cos^2 \omega_c t = \frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)$.

There is no component around f_c. Hence, this system would not give the desired output.

Problem 3. Consider an AM transmitter.

Recall that $m_p = \max_t |m(t)|$

For $m(t) = \alpha\cos(10\pi t)$, $m_p = |\alpha|$.

(a) Suppose the message is $m(t) = 4\cos(10\pi t)$ and the transmitted signal is

$$x_{\mathrm{AM}}(t) = A\cos(100\pi t) + m(t)\cos(100\pi t).$$

Find the value of A which yields the modulation index in each part below.

(i) $\mu = 50\%$ $A = \frac{4}{0.5} = 8$

(ii) $\mu = 100\%$ $A = \frac{4}{1} = 4$

(iii) $\mu = 150\%$ $A = \frac{4}{1.5} = \frac{4}{3/2} = \frac{8}{3}$

$\mu = \frac{m_p}{A} = \frac{|\alpha|}{A} = \frac{\alpha}{A}$ Let's consider only $\alpha > 0$ here.

amplitude of the carrier

Here, $\alpha = 4 \Rightarrow A = \frac{\alpha}{\mu} = \frac{4}{\mu}$

(b) Suppose the message is $m(t) = \alpha\cos(10\pi t)$ and the transmitted signal is

$$x_{\mathrm{AM}}(t) = 4\cos(100\pi t) + m(t)\cos(100\pi t).$$

Find the value of α which yields the modulation index in each part below.

(i) $\mu = 50\%$ $\alpha = 4 \times 0.5 = 2$

(ii) $\mu = 100\%$ $\alpha = 4 \times 1 = 4$

(iii) $\mu = 150\%$ $\alpha = 4 \times 1.5 = 4 \times \frac{3}{2} = 6$

Here, $A = 4 \Rightarrow \alpha = A\mu = 4\mu$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 4 (M2011Q5). In this question, you are provided with a partial proof of an important result in the study of Fourier transform. Your task is to figure out the quantities/-expressions inside the boxes labeled a,b,c, and d.

We start with a function $g(t)$. Then, we define $x(t) = \sum_{\ell=-\infty}^{\infty} g(t - \ell T)$. It is a sum that involves $g(t)$. What you will see next is our attempt to find another expression for $x(t)$ in terms of a sum that involves $G(f)$.

To do this, we first write $x(t)$ as $x(t) = g(t) * \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T)$. Then, by the convolution-in-time property, we know that $X(f)$ is given by

$$X(f) = G(f) \times \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \delta\left(f + \boxed{\qquad b \qquad}\right)$$

We can get $x(t)$ back from $X(f)$ by the inverse Fourier transform formula: $x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df$. After plugging in the expression for $X(f)$ from above, we get

$$x(t) = \int_{-\infty}^{\infty} e^{j2\pi ft} G(f) \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \delta\left(f + \boxed{\qquad b \qquad}\right) df$$

$$= \boxed{\quad a \quad} \int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} e^{j2\pi ft} G(f)\, \delta\left(f + \boxed{\qquad b \qquad}\right) df.$$

By interchanging the order of summation and integration, we have

$$x(t) = \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi ft} G(f)\, \delta\left(f + \boxed{\qquad b \qquad}\right) df.$$

We can now evaluate the integral via the sifting property of the delta function and get

$$x(t) = \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} e^{\boxed{\quad c \quad}} G\left(\boxed{\quad d \quad}\right).$$

(a) and (b) Recall that $\sum_n \delta(t - nT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0} \sum_k \delta(f - kf_0)$ where $f_0 = \frac{1}{T_0}$.

In this question, this property is applied to $\sum_\ell \delta(t - \ell T)$ to get $\sum_\ell \delta(t - \ell T) \xrightarrow{\mathcal{F}} \frac{1}{T} \sum_\ell \delta(f - \frac{\ell}{T})$

So, by the convolution-in-time rule, we have $x(t) \xrightarrow{\mathcal{F}} G(f) \times \frac{1}{T} \sum_\ell \delta(f + (-\frac{\ell}{T}))$

(c) and (d) The integral under consideration is $\int_{-\infty}^{\infty} \underbrace{e^{j2\pi ft} G(f)}_{\text{call this } b(f)} \delta(t - \frac{\ell}{T})\, df$

By the sifting property of δ-function,

$$\int_{-\infty}^{\infty} b(f)\, \delta(f - \frac{\ell}{T})\, df = b(\frac{\ell}{T}) = e^{j2\pi \frac{\ell}{T} t} G(\frac{\ell}{T})$$

Summary: $a = \frac{1}{T}$, $b = -\frac{\ell}{T}$, $c = j2\pi \frac{\ell}{T} t$, $d = \frac{\ell}{T}$

Problem 5. Would the scheme in Problem 2 work if the carrier generator output were $\cos^n \omega_c t$ for any integer $n \geq 2$?

As in Q2, we need to expand $\cos^n(x)$ into a linear combination of $\cos(kx)$.

This is a straight-forward application of the Euler's formula:

$$\cos^n x = \left(\frac{e^{jx}+e^{-jx}}{2}\right)^n = \frac{1}{2^n}\left(e^{jx}+e^{-jx}\right)^n$$

Now, apply the binomial theorem: $(A+B)^n = \sum_{k=0}^{n}\binom{n}{k}A^k B^{n-k}$. We get

$$\cos^n x = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{jxk}e^{-jx(n-k)} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{jx(2k-n)}$$

Recall that $\binom{n}{n-k} = \binom{n}{k}$ and note that $2(n-k)-n = n-2k = -(2k-n)$.

So, for every $k=c$, there would be another term $k=n-c$ to pair with. This gives

$$\binom{n}{c}e^{jx(2c-n)} + \underbrace{\binom{n}{n-c}e^{jx(2(n-c)-n)}}_{\binom{n}{c}e^{-jx(2c-n)}} = \binom{n}{c}\times 2\cos\left((2c-n)x\right)$$

This always happens except when the two terms are actually the same term which occurs when $k=n-k$ or, equivalently, $k=\frac{n}{2}$.

In which case, $\binom{n}{k}e^{jx(2k-n)} = \binom{n}{n/2} = \binom{n}{n/2}\cos(0x)$

From the analysis above, we see that $\cos^n(x)$ can be expanded into a linear combination of the cosine.

In particular, $\cos^n(2\pi f_c t)$ can be written as a linear combination of the cosines $\cos(2\pi(2k-n)f_c t)$.

Now, consider $m(t)\cos^n(2\pi f_c t)$. We want to use BPF to extract the content around $\pm f_c$. The content will be there if and only if there is a $\cos(2\pi f_c t)$ term in the expansion of $\cos^n(2\pi f_c t)$.

This happens if and only if there is a k value that makes $2k-n=1$.

For a given n, this k value is $k=\frac{n+1}{2}$.

Note that, from the binomial expansion, k must be an integer between 0 and n.

So, **n must be odd** number to give an integer-valued k.

Problem 6. Consider the basic DSB-SC transceiver with time-delay channel presented in class. Recall that the input of the receiver is

$$x(t-\tau) = m(t-\tau)\sqrt{2}\cos\left(\omega_c (t-\tau)\right)$$

$$= m(t-\tau)\sqrt{2}\cos(\omega_c t - \underbrace{\omega_c \tau}_{\equiv \phi})$$

where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$. We also assume that $f_c \gg B$.

(a) Suppose that, at the receiver, we multiply by $\sqrt{2}\cos((\omega_c t) - \theta)$ instead of $\sqrt{2}\cos(\omega_c t)$ as illustrated in Figure 5.3. Assume

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

Find $\hat{m}(t)$ (the output of the LPF).

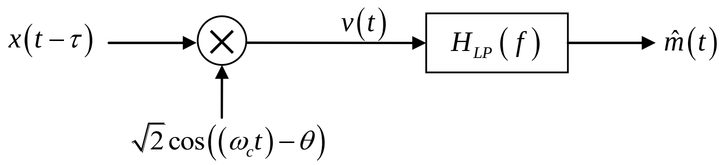


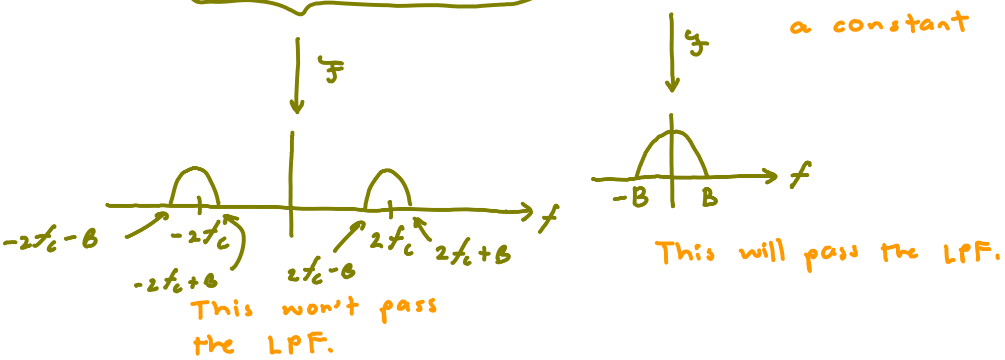
Figure 5.3: Receiver for Problem 6a

$$v(t) = x(t-\tau) \times \sqrt{2}\cos(\omega_c t - \theta)$$

$$= 2\, m(t-\tau)\cos(\omega_c t - \phi)\cos(\omega_c t - \theta)$$

$$= m(t-\tau)\Big(\cos(2\omega_c t - \phi - \theta) + \cos(\theta - \phi)\Big)$$

$$= m(t-\tau)\cos(2\omega_c t - \phi - \theta) + m(t-\tau)\underbrace{\cos(\theta - \phi)}_{\text{a constant}}$$



$$\hat{m}(t) = m(t-\tau)\cos(\theta - \phi) = m(t-\tau)\cos(\theta - \omega_c \tau).$$

Recall that

$r(t) = \frac{1}{2} + \frac{2}{\pi}\cos\omega_c t - \frac{2}{\pi}\times\frac{1}{3}\cos 3\omega_c t + \ldots = \sum_{k=0}^{\infty} a_k \cos(k\omega_c t)$

where $a_0 = \frac{1}{2}$, $a_1 = \frac{2}{\pi}$, $a_2 = 0$, $a_3 = \frac{2}{\pi}\times\frac{1}{3}$, ...

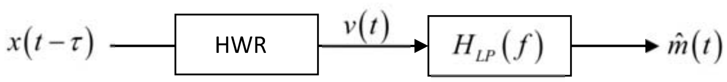


Figure 5.4: Receiver for Problem 6b

(b) Use the same assumptions as part (a). However, at the receiver, instead of multiplying by $\sqrt{2}\cos((\omega_c t) - \theta)$, we pass $x(t-\tau)$ through a half-wave rectifier (HWR) as shown in Figure 5.4b.

Make an extra assumption that $m(t) \geq 0$ for all time t and that the half-wave rectifier input-output relation is described by a function $f(\cdot)$:

$$f(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Find $\hat{m}(t)$ (the output of the LPF).

$v(t) = \underbrace{x(t-\tau)}\times r(t-\tau)$, where $r(t) = 1[\cos(2\pi f_c t) \geq 0]$

↳ Because $m(t-\tau)$ is always ≥ 0, the sign of $x(t-\tau)$ only depends on $\cos(\omega_c(t-\tau))$, which is simply a shifted version of $\cos(\omega_c t)$.

All of the analysis is the same as what was presented in class except that we now have a time shift of amount τ.

$v(t) = x(t-\tau)\times r(t-\tau)$

$= m(t-\tau)\sqrt{2}\cos(\omega_c(t-\tau))\sum_{k=0}^{\infty} a_k\cos(k\omega_c(t-\tau)) = \sqrt{2}\, m(t-\tau)\sum_{k=0}^{\infty} a_k\cos(\omega_c(t-\tau))\cos(k\omega_c(t-\tau))$

$= \sqrt{2}\, m(t-\tau)\sum_{k=0}^{\infty}\frac{1}{2}a_k\left(\cos((k-1)\omega_c(t-\tau)) + \cos((k+1)\omega_c(t-\tau))\right)$

So, $v(t)$ will be a linear combination of signals of the form

$\sqrt{2}\times\frac{1}{2}\times a_k m(t-\tau)\cos(n\omega_c(t-\tau))$

n ↳ $k-1$ or $k+1$

We know that

the spectrum of $m(t)\cos(n\omega_c t)$ is the spectrum of $m(t)$ shifted to $\pm 2\pi f_c\times n$ and scaled by $\frac{1}{2}$.

The time shift results in an extra factor of $e^{-j2\pi f\tau}$ which does not affect the location of the spectrum.

The only part of $v(t)$ that will pass through the LPF would be the one that is centered around 0 Hz (DC).

This corresponds to the case when $n = 0$

$\llcorner$ $k-1$ or $k+1$

The corresponding k is $k = 1$ or -1.

$\llcorner$ not in the summation.

Therefore, $\hat{m}(t) = \sqrt{2} \times \frac{1}{2} \times a_1 \times m(t-\tau)$. For HWR, $a_1 = \frac{2}{\pi}$. Hence, $\hat{m}(t) = \frac{\sqrt{2}}{\pi} m(t-\tau)$.

Problem 7 (M2011Q7). Suppose $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to W, i.e., $|M(f)| = 0$ for $|f| > W$. Consider a DSB-SC transceiver shown in Figure 5.5.

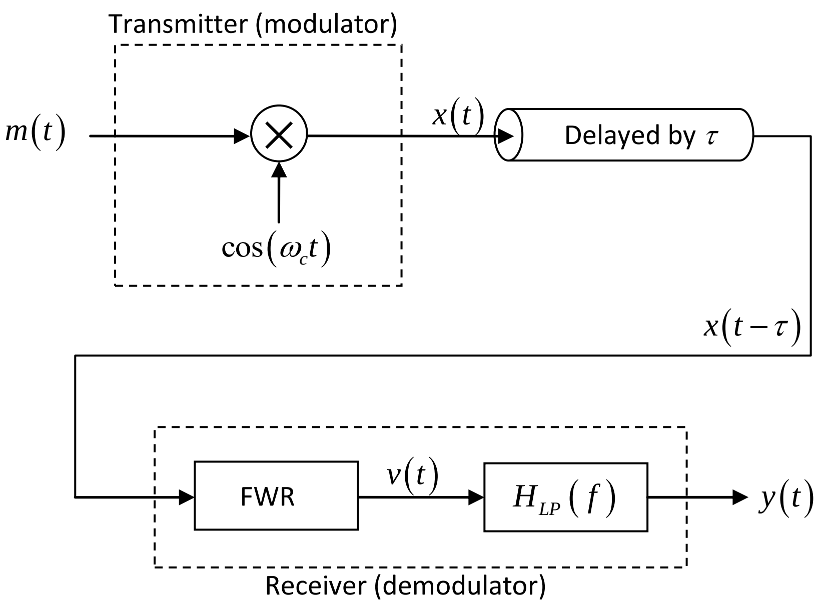


Figure 5.5: A DSB-SC transceiver

Also assume that $f_c \gg W$ and that $H_{LP}(f) = \begin{cases} 1, & |f| \le W \\ 0, & \text{otherwise.} \end{cases}$

Make an extra assumption that $m(t) \ge 0$ for all time t and that the full-wave rectifier (FWR) input-output relation is described by a function $f_{FWR}(\cdot)$: Here, the input is $x(t-\tau)$.

So, the output is

$$f_{FWR}(x) = \begin{cases} x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

$$v(t) = \begin{cases} x(t-\tau), & x(t-\tau) \ge 0 \\ -x(t-\tau), & x(t-\tau) < 0. \end{cases}$$

(a) Recall that the **half**-wave rectifier input-output relation is described by a function $f_{HWR}(\cdot)$: $f_{HWR}(x) = \begin{cases} x, & x \ge 0, \\ 0, & x < 0. \end{cases}$ We have seen in Problem 6b that when the receiver uses **half**-wave rectifier,

$$v(t) = x(t-\tau) \times g_{HWR}(t-\tau)$$

where $g_{HWR}(t) = 1[\cos(\omega_c t) \geq 0]$. The ON-OFF function.

(i) The receiver in this question uses **full**-wave rectifier. Its $v(t)$ can be described in a similar manner; that is

$$v(t) = x(t-\tau) \times g_{FWR}(t-\tau).$$

Find $g_{FWR}(t)$. Hint: $g_{FWR}(t) = c_1 \times g_{HWR}(t) + c_2$ for some constants c_1 and c_2. Find these constants.

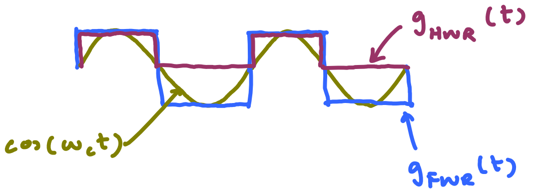
The nonnegativity of $m(t)$ means that the sign of $x(t-\tau) = m(t-\tau)\cos(\omega_c(t-\tau))$ will depend only on $\cos(\omega_c(t-\tau))$.

Therefore, $v(t) = \begin{cases} x(t-\tau), & \cos(\omega_c(t-\tau)) \geq 0 \\ -x(t-\tau), & \cos(\omega_c(t-\tau)) < 0 \end{cases} = x(t-\tau) \times g_{FWR}(t-\tau)$

where $g_{FWR}(t-\tau) = \begin{cases} 1, & \cos(\omega_c(t-\tau)) \geq 0 \\ -1, & \cos(\omega_c(t-\tau)) < 0. \end{cases}$

In other words, $g_{FWR}(t) = \begin{cases} 1, & \cos(\omega_c t) \geq 0 \\ -1, & \cos(\omega_c t) < 0. \end{cases}$

It is easier to find c_1 and c_2 via the plots of g_{FWR} and g_{HWR}.



From the plots, we have $g_{FWR}(t) = 2g_{HWR}(t) - 1$. Therefore, $c_1 = 2$ and $c_2 = -1$

(ii) Recall that the Fourier series expansion of $g_{HWR}(t)$ is given by

$$g_{HWR}(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\omega_c t - \frac{1}{3}\cos 3\omega_c t + \frac{1}{5}\cos 5\omega_c t - \frac{1}{7}\cos 7\omega_c t + \dots\right).$$

Find the Fourier series expansion of $g_{FWR}(t)$.

$$g_{HWR}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}\cos((2k-1)\omega_c t)$$

Therefore, $g_{FWR}(t) = 2g_{HWR}(t) - 1 = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos((2k-1)\omega_c t)$.

(b) Find $y(t)$ (the output of the LPF).

$y(t) = \text{LPF}\{v(t)\}$ where $v(t) = m(t-\tau)\cos(\omega_c(t-\tau))\, g_{FWR}(t-\tau)$.

Let's first consider $v(t+\tau) = m(t)\cos(\omega_c t)\, g_{FWR}(t)$.

$$v(t+\tau) = m(t)\cos(\omega_c t)\,\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos((2k-1)\omega_c t)$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}\Big(m(t)\cos((2k-2)\omega_c t) + m(t)\cos(2k\,\omega_c t)\Big)$$

In freq. domain, these terms will be replica of $M(f)$ shifted to various frequencies.

The only term that shifts to DC is this one at $k=1$.

So, $y(t) = \text{LPF}\{v(t)\} = \frac{2}{\pi} m(t-\tau)$.