Name Prapun ID3 5 5 5

ECS 332: Principles of Communications **2017/1**

HW 4 — Due: Not Due **Solution**

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Consider the DSB-SC modem with no channel impairment shown in Figure 4.1. Suppose that the message is band-limited to $B = 3$ kHz and that $f_c = 100$ kHz.

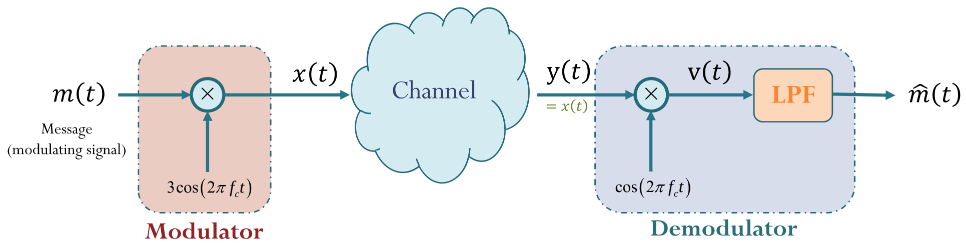
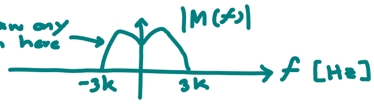


Figure 4.1: DSB-SC modem with no channel impairment

(a) Specify the frequency response $H_{LP}(f)$ of the LPF so that $\hat{m}(t) = m(t)$.

$M(f)$ is assumed to be band-limited to $B = 3$ kHz. Therefore, $M(f) = 0$ for $|f| > 3$ kHz :



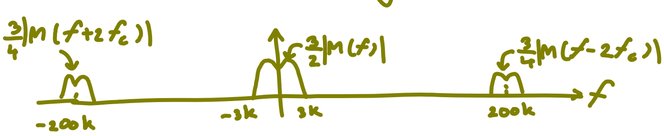
$$v(t) = y(t)\times\cos(2\pi f_c t) \underset{y(t)=x(t)}{=} x(t)\times\cos(2\pi f_c t) \underset{x(t)=m(t)\times 3\cos(2\pi f_c t)}{=} 3m(t)\cos^2(2\pi f_c t) = 3m(t)\times\frac{1}{2}\left(1+\cos(2\pi(2f_c)t)\right)$$

$$= \frac{3}{2}m(t) + \frac{3}{2}m(t)\cos(2\pi(2f_c)t)$$

$$V(f) = \frac{3}{2}M(f) + \frac{3}{4}M(f-2f_c) + \frac{3}{4}M(f+2f_c).$$

Here, $f_c = 100$ kHz

Given the picture of $|M(f)|$ above, we now draw the corresponding $|V(f)|$:



To eliminate the terms $\frac{3}{4}M(f-2f_c)$ and $\frac{3}{4}M(f+2f_c)$, we set $H_{LP}(f) = 0$ for $|f| > 200k - 3k = 197$ kHz.

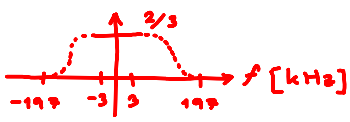
To allow $\frac{3}{2}M(f)$ to pass through, we set $H_{LP}(f) = c$ for $|f| < 3$kHz for some constant c.

With such $H_{LP}(f)$, we get $\hat{m}(t) = c\times\frac{3}{2}m(t)$.

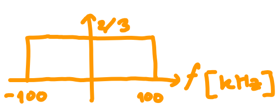
Because we need $\hat{m}(t) = m(t)$,

we have to set $\frac{3}{2}c = 1 \Rightarrow c = \frac{2}{3}$.

$$H_{LP}(f) = \begin{cases} \frac{2}{3}, & |f| \le 3\text{kHz}, \\ 0, & |f| \ge 197 \text{ kHz}, \\ \text{"any"}, & \text{otherwise}. \end{cases}$$



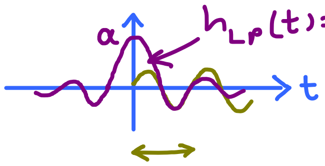
An example would be $H_{LP}(f) = \begin{cases} 2/3, & |f| \le 100 \text{ kHz}, \\ 0, & \text{otherwise} \end{cases}$



(b) Suppose the impluse response $h_{LP}(t)$ of the LPF is of the form $\alpha\,\mathrm{sinc}(\beta t)$. Find the constants α and β such that $\hat{m}(t) = m(t)$.

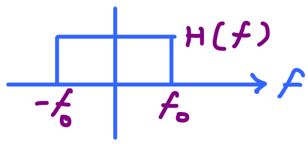
Observation:
The frequency of the sine wave in the sinc function fives the boundaries of the rectangular function in another domain.

$h_{LP}(t) = \alpha\,\mathrm{sinc}(\beta t)$ $= 2\pi\left(\frac{\beta}{2\pi}\right)t$



freq. of the sine wave

So, period = $\frac{2\pi}{\beta}$



period = $\frac{2\pi}{\beta}$

$\frac{1}{\text{width}} = \frac{\text{period}}{2} = \frac{\pi}{\beta}$
(of the rectangular pulse)

width = $\frac{\beta}{\pi}$

$f_0 = \frac{\beta}{2\pi}$

width = $2f_0$

Area = $h_{LP}(0) = \alpha$

height × $2f_0 = \alpha$

height $= \frac{\alpha}{2f_0} = \frac{\alpha}{2\times\frac{\beta}{2\pi}} = \pi\frac{\alpha}{\beta}$

the gain of the LPF

From the previous part, we need $3 \leq f_0 < 197$ kHz.

$$\text{gain} = \frac{2}{3} \Rightarrow \pi\frac{\alpha}{\beta} = \frac{2}{3} \Rightarrow \alpha = \frac{2\beta}{3\pi}.$$

So, first we choose f_0. Then, we have $\beta = 2\pi f_0$ and $\alpha = \frac{2\beta}{3\pi}$.

An example would be $f_0 = 100$ kHz $\Rightarrow \beta = 2\times10^5\pi$ rad/s and $\alpha = \frac{2}{3\cancel{\pi}}\times 2\times10^5\cancel{\pi} = \frac{4}{3}\times10^5$

Alternatively, one may use $f_0 = B = 3$ kHz. Then, $\beta = 2\pi f_0 = 2\pi\times3k = 6\times10^3\pi$ rad/s

$$\alpha = \frac{2}{3\pi}\beta = \frac{2}{\cancel{3}\cancel{\pi}}\times\cancel{6}^{\,2}\times10^3\cancel{\pi} = 4000.$$

Problem 2. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 4.2. Note that V and T_b are some positive constants. Your answers should be given in terms of them.

(a) Find the energy in each signal.

$$E_{s_1} = \int_{-\infty}^{\infty}|s_1(t)|^2dt = \int_0^{T_b}V^2dt = V^2T_b.$$

$$E_{s_2} = \int_{-\infty}^{\infty}|s_2(t)|^2dt = \int_0^{T_b}V^2dt = V^2T_b.$$

same (total) energy

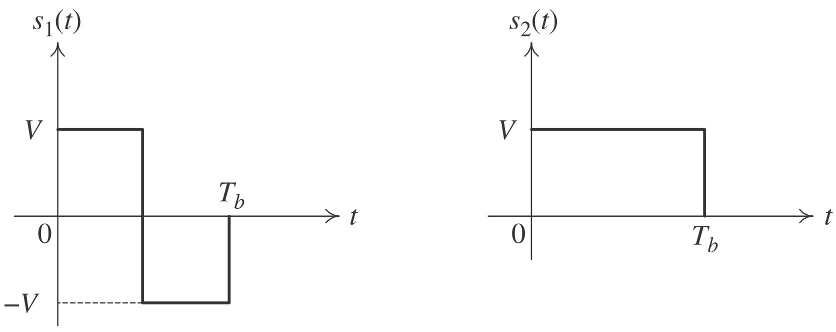


Figure 4.2: Signal set for Question 2

(b) Are they energy signals?

Because V and T_b are positive constants, we know that $V^2 T_b$ is positive and finite.
Therefore, $0 < E_{s_1}, E_{s_2} < \infty$. Hence, both s_1 and s_2 are energy signals $\Rightarrow$ Yes

(c) Are they power signals?

No. Because they are energy signals, they can not be power signals.

(d) Find the (average) power in each signal.

All energy signals have 0 (average) power.

[See the comment 1 at the end]

(e) Are the two signals $s_1(t)$ and $s_2(t)$ orthogonal?

$$\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t)\, s_2(t)\, dt = \int_{0}^{T_b/2} V \times V\, dt + \int_{T_b/2}^{T_b} (-V)\times(V)\, dt = V^2 \frac{T_b}{2} - V^2 \frac{T_b}{2} = 0.$$

Because $\langle s_1, s_2 \rangle = 0$, we know that s_1 and s_2 are orthogonal.

Problem 3. (Power Calculation) For each of the following signals $g(t)$, find (i) its corresponding power $P_g = \left\langle |g(t)|^2 \right\rangle$, (ii) the power $P_x = \left\langle |x(t)|^2 \right\rangle$ of $x(t) = g(t)\cos(10t)$, and (iii) the power $P_y = \left\langle |y(t)|^2 \right\rangle$ of $y(t) = g(t)\cos(50t)$

(a) $g(t) = 3\cos(10t + 30°)$.

Assume $f_0 \neq 0$ A = 3.

(a.i) $g(t) = A\cos(2\pi f_0 t + \theta) \Rightarrow P_g = \frac{|A|^2}{2} \Rightarrow P_g = \frac{|3|^2}{2} = \frac{9}{2} = 4.5.$

(a.ii) $x(t) = g(t)\cos(10t) = \left(3\cos(10t+30^\circ)\right)\left(\cos(10t)\right) \overset{\text{product-to-sum formula}}{=} \frac{3}{2}\left(\cos(20t+30^\circ) + \underbrace{\cos(30^\circ)}_{\text{a constant}}\right)$

nonoverlapping in the freq. domain

comment 3

$P_x = \left(\frac{3}{2}\right)^2\left(\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2\right) = \frac{9}{4}\left(\frac{1}{2}+\frac{3}{4}\right) = \frac{9}{4}\left(\frac{5}{4}\right) = \frac{45}{16} \approx 2.813$

[See comment 2 and comment 4 at the end]

(a.iii) $y(t) = g(t)\cos(50t) = \left(3\cos(10t+30^\circ)\right)\left(\cos(50t)\right) \overset{\text{product-to-sum formula}}{=} \frac{3}{2}\left(\cos(60t+30^\circ) + \cos(40t-30^\circ)\right)$

different freq.

$P_y = \left(\frac{3}{2}\right)^2\left(\frac{1}{2}+\frac{1}{2}\right) = \frac{9}{4}\times 1 = \frac{9}{4}$

[See comment 5 at the end]

[See comment 6 at the end]

(b) $g(t) = 3\cos(10t+30^\circ) + 4\cos(10t+120^\circ)$. (Hint: First, use phasor form to combine the two components into one sinusoid.)

(b.i) $g(t) = 3\cos(10t+30^\circ) + 4\cos(10t+120^\circ) = \mathrm{Re}\left\{\underbrace{\left(3\angle 30^\circ + 4\angle 120^\circ\right)}_{\approx 0.5981+4.9641j \approx 5\angle 83.13^\circ} e^{j10t}\right\}$

$= 5\cos(10t + 83.13^\circ)$

Note that we do not need the phase 83.13° to calculate the average power. Also, we can get the magnitude "5" simply by noticing the 90° difference between $3\angle 30^\circ$ and $4\angle 120^\circ$.

$\sqrt{3^2+4^2} = 5$

$P_g = 5^2 \times \frac{1}{2} = \frac{25}{2} = 12.5$

(b.ii) $x(t) = g(t)\cos(10t) = 5\cos(10t+83.13^\circ)\cos(10t)$

$= \frac{5}{2}\left(\cos(20t+83.13^\circ) + \cos(83.13^\circ)\right)$

$P_x = \left(\frac{5}{2}\right)^2\left(\frac{1}{2} + \cos^2(83.13^\circ)\right) = \frac{25}{8}\left(1 + 2\cos^2(83.13^\circ)\right) \approx 3.214$

(b.iii) Note that $G(f)$ is still at $\pm\frac{10}{2\pi}$ as in part (a.iii).

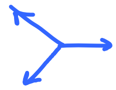
Therefore, $G\left(f-\frac{50}{2\pi}\right)$ and $G\left(f+\frac{50}{2\pi}\right)$ still do not overlap in the freq. domain.

$P_y = \frac{1}{2}P_g = \frac{25}{4} = 6.25$

(c) $g(t) = 3\cos(10t) + 3\cos(10t + 120°) + 3\cos(10t + 240°)$

(c.i) Look at the three components of $g(t)$ in their phasor representation.

We have $3\angle 0° + 3\angle 120° + 3\angle 240° = 0$



clear when you draw the three vectors

Therefore, $g(t) = 0$. Hence, $P_g = 0$.

(c.ii) $x(t) = 0 \Rightarrow P_x = 0$

(c.iii) $y(t) = 0 \Rightarrow P_y = 0$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 4. This question starts with a ***square-modulator*** for DSB-SC . Then, the use of the square-operation block is further explored on the receiver side of the system. [Doerschuk, 2008, Cornell ECE 320]

(a) Let $x(t) = A_c m(t)$ where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$. Consider the block diagram shown in Figure 4.3.

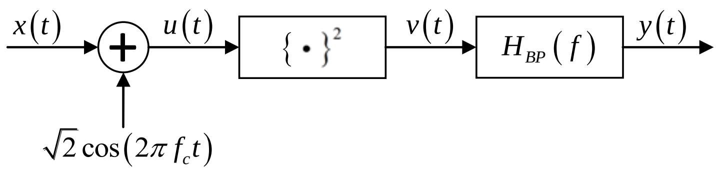


Figure 4.3: Block diagram for Problem 4a

Assume $f_c \gg B$ and

$$H_{BP}(f) = \begin{cases} 1, & |f - f_c| \le B \\ 1, & |f + f_c| \le B \\ 0, & \text{otherwise.} \end{cases}$$

The block labeled "$\{\cdot\}^2$" has output $v(t)$ that is the square of its input $u(t)$:

$$v(t) = u^2(t).$$

Find $y(t)$.

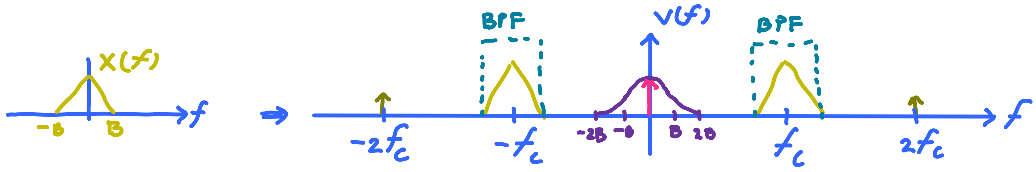
$x(t) = A_c m(t) \xrightarrow{\mathcal{F}} X(f) = A_c M(f)$. So, $X(f)$ is also bandlimited to B.

$u(t) = x(t) + \sqrt{2}\cos(\omega_c t)$ $\quad \omega_c = 2\pi f_c$

$v(t) = u^2(t) = (x(t) + \sqrt{2}\cos(\omega_c t))^2 = x^2(t) + 2\sqrt{2}x(t)\cos(\omega_c t) + \underbrace{2\cos^2(\omega_c t)}_{1+\cos(2\omega_c t)}$

$= 1 + x^2(t) + 2\sqrt{2}x(t)\cos(2\pi f_c t) + \cos(2\pi(2f_c)t)$

For example,



Note 1: $x^2(t) \xrightarrow{\mathcal{F}} X(f) * X(f)$. So, $x^2(t)$ is bandlimited to $2B$.
Because $f_c \gg B$, the spectrum of $x^2(t)$ will not be in the passband of the BPF which centers around f_c.

Note 2: The term $\cos(2\omega_c t)$ is at frequency $2f_c$ which again is outside the passband of the BPF.

Therefore, only the term $2\sqrt{2}x(t)\cos(2\pi f_c t)$ will survive the BPF.

$y(t) = BPF\{v(t)\} = 2\sqrt{2}x(t)\cos\omega_c t = 2\sqrt{2}A_c m(t)\cos\omega_c t$

[See comment 7 at the end]

(b) The block diagram in part (a) provides a nice implementation of a modulator because it may be easier to build a squarer than to build a multiplier. Based on the successful use of a squaring operation in the modulator, we decide to use the same squaring operation in the demodulator. Let

$$x(t) = A_c m(t)\sqrt{2}\cos(2\pi f_c t)$$

where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$. Again, assume $f_c \gg B$ Consider the block diagram shown in Figure 4.4.

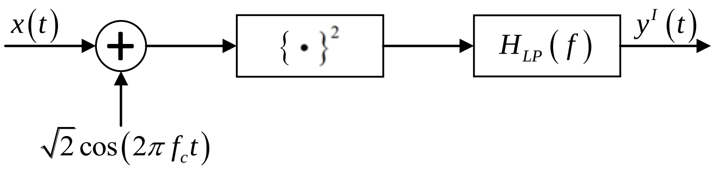


Figure 4.4: Block diagram for Problem 4b

Use

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

Find $y^I(t)$. Does this block diagram work as a demodulator; that is, is $y^I(t)$ proportional to $m(t)$?

$$v(t) = \left(x(t) + \sqrt{2}\cos(\omega_c t)\right)^2 = 2\cos^2(\omega_c t)\left(A_c m(t) + 1\right)^2$$

$$= 1 + \cos(2\omega_c t)\underbrace{\left(A_c^2 \underbrace{m^2(t)}_{\text{band-limited to } 2B} + 1 + 2A_c \underbrace{m(t)}_{\text{band-limited to } B}\right)}_{\text{Define this part as } g(t)} = g(t) + \overset{\text{LPF}}{\underset{0}{\cancel{g(t)\cos(2\omega_c t)}}}$$

$g(t)\cos(2\omega_c t)$ is centered @ $2f_c$ and therefore will not pass though the LPF.

$$y^I(t) = \text{LPF}\{v(t)\} = \text{LPF}\{g(t)\} = 1 + 2A_c m(t) + \text{LPF}\left\{A_c^2 m^2(t)\right\}$$

This term has spectrum beyond $\pm B$. So, only a portion of it will pass through the LPF.

$y^I(t)$ is not proportional to $m(t)$.

Hence, this block diagram does not work as a demodulator.

(c) Due to the failure in part (b), we have to think hard and it seems natural to consider also the block diagram with cos replaced by sin. Let

$$x(t) = A_c m(t)\sqrt{2}\cos(2\pi f_c t)$$

where $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ is bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$ as in part (b). Again, assume $f_c \gg B$ Consider the block diagram shown in Figure 4.5.

As in part (b), use

$$H_{LP}(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

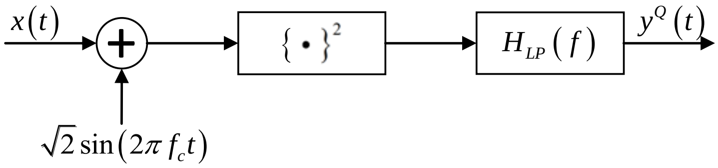
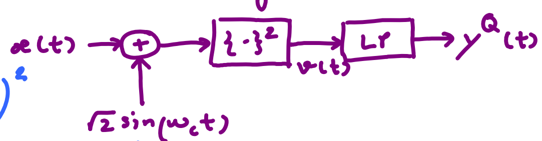


Figure 4.5: Block diagram for Problem 4c

Find $y^Q(t)$.

New block diagram



Let $x(t) = A_c m(t)\sqrt{2}\cos(\omega_c t)$ as in part (b).

$$v(t) = \left(x(t) + \sqrt{2}\sin(\omega_c t)\right)^2 = 2\left(A_c m(t)\cos(\omega_c t) + \sin(\omega_c t)\right)^2$$

$$= 2\left(A_c^2 m^2(t)\cos^2(\omega_c t) + A_c m(t)\cos(\omega_c t)\sin(\omega_c t) + \sin^2(\omega_c t)\right)$$

$2\cos\beta\sin\beta = \sin(2\beta)$

$$= 2\left(A_c^2 m^2(t)\cos^2(\omega_c t) + \underbrace{\sin^2(\omega_c t)}_{= 1 - \cos^2\omega_c t}\right) + A_c m(t)\sin(2\omega_c t)$$

$$= 2\left(\left(A_c^2 m^2(t) - 1\right)\cos^2(\omega_c t) + 1\right) + A_c m(t)\sin(2\omega_c t)$$

$$= 2 + \left(A_c^2 m^2(t) - 1\right)\left(1 + \underbrace{\cos(2\omega_c t)}_{\text{LPF} \to 0}\right) + A_c m(t)\underbrace{\sin(2\omega_c t)}_{\text{LPF} \to 0}$$

$$y^Q(t) = 2 + \text{LPF}\{A_c^2 m^2(t)\} - 1 = \text{LPF}\{A_c^2 m^2(t)\} + 1$$

The output alone is far from being proportional to $m(t)$.
So, this block diagram also does not work as a demodulator.

(d) Use the results from parts (b) and (c). Draw a block diagram of a ***successful*** DSB-SC demodulator using squaring operations instead of multipliers.

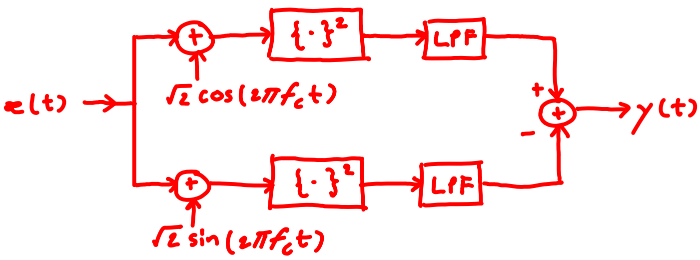
Observe that

$y^I(t) - y^Q(t) = 2A_c m(t)$

($y^I(t)$ from (b), $y^Q(t)$ from (c))

which is the desired output of a successful DSB-SC demodulator.

Hence, the following block diagram would work:



Problem 5 (Cube modulator). Consider the block diagram shown in Figure 4.6 where "$\{\cdot\}^3$" indicates a device whose output is the cube of its input.

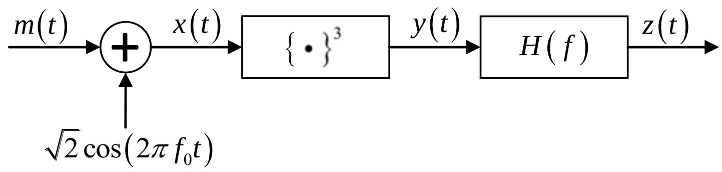


Figure 4.6: Block diagram for Problem 5. Note the use of f_0 instead of f_c.

Let $m(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} M(f)$ be bandlimited to B, i.e., $|M(f)| = 0$ for $|f| > B$.

(a) Plot an $H(f)$ that gives $z(t) = m(t)\sqrt{2}\cos(2\pi f_c t)$. What is the gain in $H(f)$? What is the value of f_c? Notice that the frequency of the cosine is f_0 not f_c. You are supposed to determine f_c in terms of f_0.

$$y(t) = \left(m(t) + \sqrt{2}\cos(2\pi f_0 t)\right)^3 = m^3(t) + 3m^2(t)\sqrt{2}\cos\omega_0 t + \underbrace{3m(t)\,2\cos^2\omega_0 t}_{\substack{= 3m(t)(1+\cos 2\omega_0 t) \\ = 3m(t) + 3m(t)\cos(2\omega_0 t)}} + \underbrace{(\sqrt{2})^3\cos^3(\omega_0 t)}_{= \frac{3}{\sqrt{2}}\cos(\omega_0 t) + \frac{1}{\sqrt{2}}\cos(3\omega_0 t)}$$

$$\left\{\begin{array}{l} 2\cos^2(\theta) = 1 + \cos(2\theta) \\ 2\cos^3(\theta) = \cos\theta + \cos\theta\cos 2\theta \\ \qquad = \cos\theta + \frac{1}{2}\cos\theta + \frac{1}{2}\cos 3\theta \\ \qquad = \frac{3}{2}\cos\theta + \frac{1}{2}\cos 3\theta \end{array}\right\}$$

We want $z(t) = m(t)\sqrt{2}\cos(\omega_c t)$. We see that the only term in $y(t)$ that has the form "constant × m × cos( )" is $3m(t)\cos(2\omega_0 t)$.

Therefore, we will center the passband to cover this part and adjust the gain to make the output the same as $z(t)$.

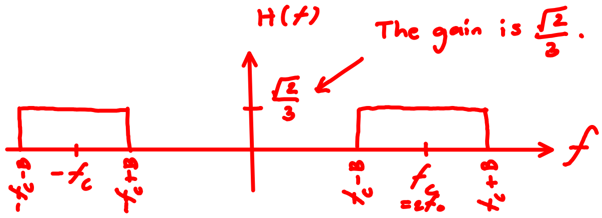
In particular, we need to make $2f_0 = f_c$. So, $f_0 = f_c/2$.

Let $H_{BP}(f) = \begin{cases} g, & |f - f_c| \le B, \\ g, & |f + f_c| \le B, \\ 0, & \text{otherwise.} \end{cases}$

Then, $z(t) = \underbrace{g \times 3}\, m(t)\cos(2\omega_0 t)$

we need $g \times 3 = \sqrt{2} \Rightarrow g = \frac{\sqrt{2}}{3}$

$\Rightarrow$



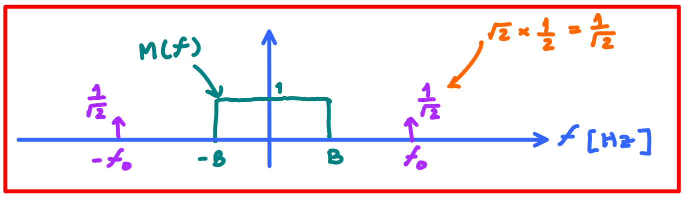
The gain is $\frac{\sqrt{2}}{3}$.

(b) Let $M(f)$ be

$$M(f) = \begin{cases} 1, & |f| \le B \\ 0, & \text{otherwise.} \end{cases}$$

(i) Plot $X(f)$.

$x(t) = m(t) + \sqrt{2}\cos(2\pi f_0 t)$



(ii) Plot $Y(f)$. Hint:

$$M(f) * M(f) = \begin{cases} 2B - |f|, & |f| \le 2B \\ 0, & \text{otherwise.} \end{cases}$$

Do not attempt to make an accurate plot or calculation for the Fourier transform of $m^3(t)$.

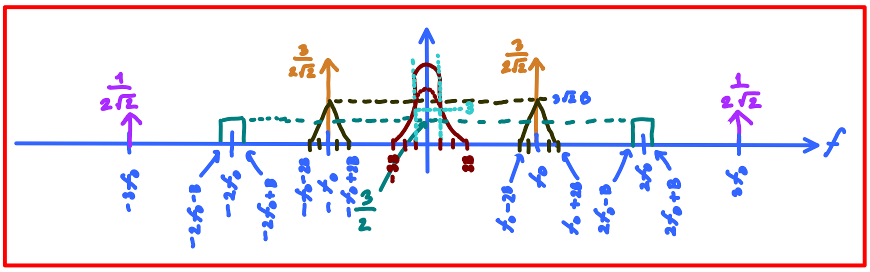
From (a), we have

$$y(t) = m^3(t) + 3m(t) + 3\sqrt{2}\, m^2(t)\cos(\omega_0 t) + 3m(t)\cos(2\omega_0 t) + \frac{1}{\sqrt{2}}\cos(3\omega_0 t) + \frac{3}{\sqrt{2}}\cos(\omega_0 t)$$

Without trying to make an accurate plot for $m^3(t)$, we know that it is band limited to $3B$.

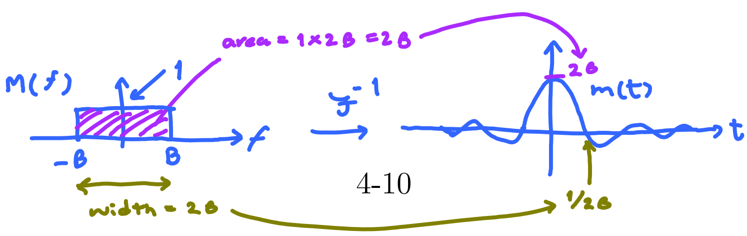
If you want to know the shape of $M(f)*M(f)*M(f)$, you can try plotting it in MATLAB using this code:

```
w = ones(1,10);
w2 = conv(w,w);
w3 = conv(w2,w);
plot(w3)
```



(iii) For your filter of part (a), plot $z(t)$.

$z(t) = m(t)\sqrt{2}\cos(2\pi f_c t)$. Therefore, to plot $z(t)$, first we need to find $m(t)$.

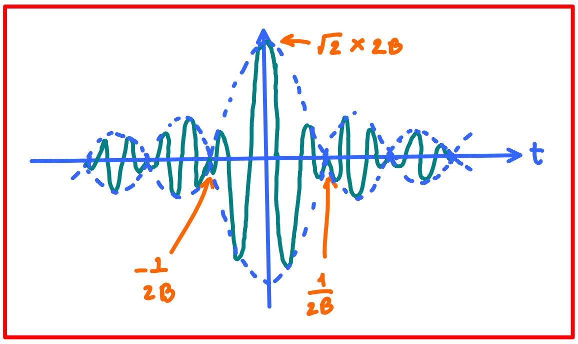


$z(t)$ is the above sinc. function multiplied by $\cos(2\pi f_c t)$.

Because $f_c \gg B$, we know that $\frac{1}{B} \gg \frac{1}{f_c}$

period of sinc inside sinc; period of cosine

So, the |sinc| function becomes the envelope of the cosine carrier.



[Doerschuk, 2008, Cornell ECE 320]

Problem 6. Consider a signal $g(t)$. Recall that $|G(f)|^2$ is called the **energy spectral density** of $g(t)$. Integrating the energy spectral density over all frequency gives the signal's total energy. Furthermore, the energy contained in the frequency band I can be found from the integral $\int_I |G(f)|^2 df$ where the integration is over the frequencies in band I. In particular, if the band is simply an interval of frequency from f_1 to f_2, then the energy contained in this band is given by

$$\int_{f_1}^{f_2} |G(f)|^2 df. \tag{4.1}$$

In this problem, assume

$$g(t) = 1[-1 \le t \le 1].$$

(a) Find the (total) energy of $g(t)$.

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} (1[-1 \le t \le 1])^2 dt = \int_{-1}^{1} 1\, dt = 2.$$

Remark: We can also try to find E_g from the freq. domain.

In part (b), we will show that $G(f) = 2\,\mathrm{sinc}(2\pi f)$.

Therefore,

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt \underset{\text{Parseval's theorem}}{=} \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} (2\,\mathrm{sinc}(2\pi f))^2 df$$

$\mu = 2\pi f$

$d\mu = 2\pi\, df$

$$= \frac{1}{2\pi} \times 4 \int_{-\infty}^{\infty} \mathrm{sinc}^2(\mu)\, d\mu \overset{\text{Ex. 2.44.6}}{=} \frac{1}{2\pi} \times 4 \times \pi = 2$$

(b) Figure 4.7 define the main lobe of a sinc pulse. It is well-known that the main lobe of the sinc function contains about 90% of its total energy. Check this fact by first

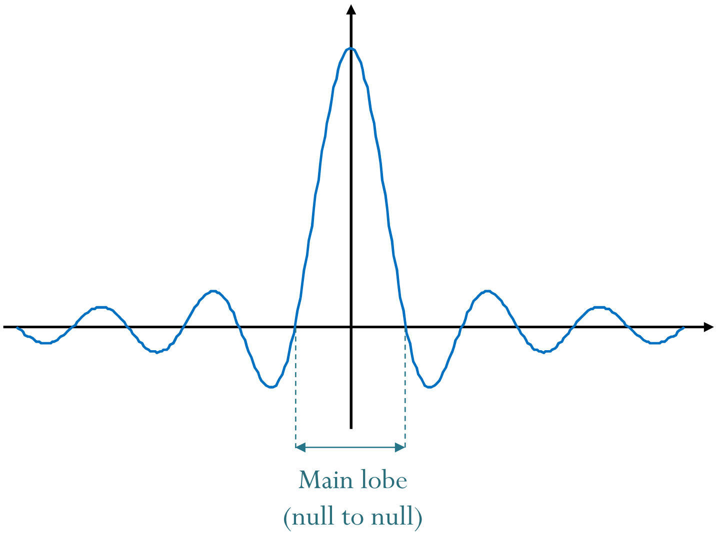


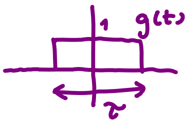
Figure 4.7: Main lobe of a sinc pulse

computing the energy contained in the frequency band occupied by the main lobe and then compare with your answer from part (a).

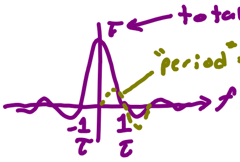
Hint: Find the zeros of the main lope. This give f_1 and f_2. Now, we can apply (4.1). `MATLAB` or similar tools can then be used to numerically evaluate the integral.

First, we need $G(f)$.

Recall that



$\xrightarrow{\mathcal{F}}$



$\tau \leftarrow$ total area in time-domain

"period" $= \frac{2}{\tau}$ $\Rightarrow$ "freq" $= \frac{\tau}{2}$ $\Rightarrow \sin(2\pi \frac{\tau}{2} f)$

$G(f) = \tau \operatorname{sinc}(\pi f \tau)$

Here, $\tau = 2$. So, $G(f) = 2\operatorname{sinc}(2\pi f)$

The main lobe occupies an interval of frequency from $f_1 = -\frac{1}{\tau} = -\frac{1}{2}$ to $f_2 = +\frac{1}{\tau} = +\frac{1}{2}$.

So, the energy contained in the band $B = [f_1, f_2]$ is given by $\int_{-1/2}^{1/2} (2\operatorname{sinc}(2\pi f))^2 \, df \approx 1.8036$ ← MATLAB

Compared with the answer from part (a), this is $\approx 90\%$ of the total energy.

(c) Suppose we want to include more energy by considering wider frequency band. Let this band be the interval $I = [-f_0, f_0]$. Find the minimum value of f_0 that allows the band to capture at least 99% of the total energy in $g(t)$.

Using MATLAB, we can look at the fraction of energy as a function of f_0.

We found that at around $f_0 \approx 5.1$, the fraction begins to exceed 99%.

Comment 1: Additional proof for Q2d:

All energy signals have 0 average power

Consider $P_g = \lim_{T\to\infty} \int_{-T/2}^{T/2} |g(t)|^2 dt.$

Note that $|g(t)|^2$ is always nonnegative. Therefore,

$$0 \le \int_{-T/2}^{T/2} |g(t)|^2 dt \le \int_{-\infty}^{\infty} |g(t)|^2 dt = E_g$$

g(t) is an energy signal; so this is a finite number.

$$0 \le \frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt \le \frac{1}{T} E_g$$

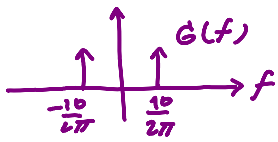
$\downarrow T\to\infty \qquad \downarrow T\to\infty$ Here, we take the limit as $T\to\infty$.

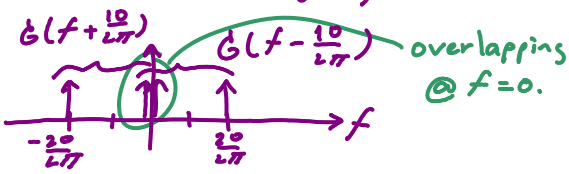
$$0 \le P_g \le 0$$

Therefore, $P_g = 0$.

Comment 2: Additional comment for Q3.a.ii:

Note that although $x(t) = g(t)\cos(2\pi f_0 t)$, we can't use $P_x = \frac{1}{2}P_g$ because $G(f-f_0)$ and $G(f+f_0)$ overlap in the frequency domain.





Comment 3: A property that we frequently use in power calculation

Let $v(t) = a\,u(t)$. Then

$$P_v = \langle |v^2(t)| \rangle = \langle |a^2 u^2(t)| \rangle = |a|^2 \langle |u^2(t)| \rangle = |a|^2 P_u$$

Comment 4: More comment for Q3.a.ii:

In general, for $x(t) = a\cos(2\pi f_0 t + \theta)\cos(2\pi f_0 t + \phi)$,

applying the product-to-sum formula gives

$$x(t) = \frac{a}{2}\left(\cos\left(2\pi(2f_0)t + \theta + \phi\right) + \cos(\theta - \phi)\right)$$

When $f_0 \ne 0$, the two cosine components do not overlap in the frequency domain. Hence, the power of their sum is the same as the sum of their power.

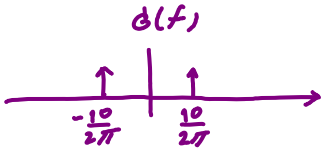
Therefore, $P_x = \left|\frac{a}{2}\right|^2 \left(\frac{1}{2} + \cos^2(\theta-\phi)\right).$

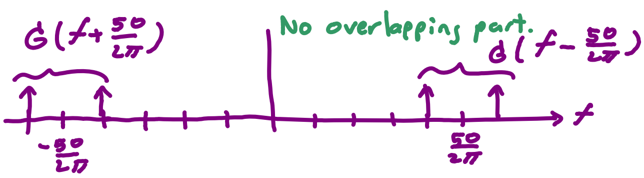
Here, $a = 3$, $\theta = 0$, $\phi = 30°$.

Therefore, $P_x = \left(\frac{3}{2}\right)^2\left(\frac{1}{2} + \cos^2(30°)\right) = \frac{9}{4}\left(\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2\right) = \frac{9}{4}\left(\frac{1}{2} + \frac{3}{4}\right) = \frac{45}{16}$

Comment 5: comment for Q3.a.iii:

Note that $P_y = \frac{1}{2}P_g$ because $G(f - \frac{50}{2\pi})$ and $G(f + \frac{50}{2\pi})$ do not overlap.





Comment 6: Alternative solution for Q3.a

(a.i) $g(t) = 3\cos(10t + 30^\circ) = \frac{3}{2}\left(e^{j(10t+30^\circ)} + e^{-j(10t+30^\circ)}\right)$

$= \frac{3}{2} e^{j30^\circ} e^{j10t} + \frac{3}{2} e^{-j30^\circ} e^{-j10t}$

$P_g = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 2 \times \frac{9}{4} = \frac{9}{2} = 4.5$

(a.ii) $y(t) = g(t)\cos(50t) = \frac{3}{2}\left(e^{j30^\circ}e^{j10t} + e^{-j30^\circ}e^{-j10t}\right)\frac{1}{2}\left(e^{j50t} + e^{-j50t}\right)$

$= \frac{3}{4}\left(e^{j30^\circ}e^{j60t} + e^{-j30^\circ}e^{j40t} + e^{j30^\circ}e^{-j40t} + e^{-j30^\circ}e^{-j60t}\right)$

All of the complex exponential functions have distinct frequencies.

$P_y = \left(\frac{3}{4}\right)^2\left(1^2 + 1^2 + 1^2 + 1^2\right) = \frac{9}{16} \times 4 = \frac{9}{4} \approx 2.25$

(a.iii) $y(t) = g(t)\cos(50t) = \frac{3}{2}\left(e^{j30^\circ}e^{j10t} + e^{-j30^\circ}e^{-j10t}\right)\frac{1}{2}\left(e^{j50t} + e^{-j50t}\right)$

$= \frac{3}{4}\left(e^{j30^\circ}e^{j60t} + e^{-j30^\circ}e^{j40t} + e^{j30^\circ}e^{-j40t} + e^{-j30^\circ}e^{-j60t}\right)$

All of the complex exponential functions have distinct frequencies.

$P_y = \left(\frac{3}{4}\right)^2\left(1^2 + 1^2 + 1^2 + 1^2\right) = \frac{9}{16} \times 4 = \frac{9}{4} \approx 2.25$

Comment 7: Remark for Q4.a

More generally, if the gain of the filter is "g" and the amplitude of the carrier is c,
then $y(t) = 2c \times g \times (\text{input signal}) \times \cos(\omega_c t)$
Here, $c = \sqrt{2}$, $g = 1$, and input signal $= A_c m(t)$.
Therefore, $y(t) = 2\sqrt{2} A_c m(t) \cos(\omega_c t)$.