Q1 Energy Calculation

Sunday, September 20, 2015 3:23 PM

(a)

$$E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$\Big\}$ same (total) energy

(b) Because V and T_b are positive constants,
we know that $V^2 T_b$ is positive and finite.
Therefore, $0 < E_{s_1}, E_{s_2} < \infty$.

Hence, both s_1 and s_2 are energy signals $\Rightarrow$ Yes

(c) No. Because they are energy signals, they can not be power signals.

(d) All energy signals have 0 (average) power.
To see this, consider $P_g = \lim_{T\to\infty} \int_{-T/2}^{T/2} |g(t)|^2 dt$.

Note that $|g(t)|^2$ is always nonnegative Therefore,

$$0 \le \int_{-T/2}^{T/2} |g(t)|^2 dt \le \int_{-\infty}^{\infty} |g(t)|^2 dt = E_g$$

$g(t)$ is an enery signal; so, this is a finite number

$$0 \le \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt \le \frac{1}{T} E_g$$

$\downarrow T\to\infty \qquad \downarrow T\to\infty$

Here, we take the limit as $T\to\infty$.

$$0 \le P_g \le 0$$

Therefore, $P_g = 0$.

(e) $\langle s_1, s_2 \rangle = \int_{-\infty}^{\infty} s_1(t) s_2(t)\, dt = \int_0^{T_b/2} V \times V\, dt + \int_{T_b/2}^{T_b} (-V)\times(V)\, dt = V^2 \frac{T_b}{2} - V^2 \frac{T_b}{2} = 0.$

Because $\langle s_1, s_2 \rangle = 0$, we know that s_1 and s_2 are orthogonal.

Q2 Average power in DSB-SC

Tuesday, November 13, 2012 4:10 PM

(a.i) $g(t) = 3\cos(10t + 30^\circ) = \frac{3}{2}\left(e^{j(10t+30^\circ)} + e^{-j(10t+30^\circ)}\right)$

$$= \frac{3}{2}e^{j30^\circ}e^{j10t} + \frac{3}{2}e^{-j30^\circ}e^{-j10t}$$

$$P_g = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 2 \times \frac{9}{4} = \frac{9}{2} = 4.5$$

Assume $f_0 \neq 0$

Alternatively, we know that for $g(t) = A\cos(2\pi f_0 t + \theta)$, $P_g = \frac{|A|^2}{2}$.

Here, $A = 3$. Therefore $P_g = \frac{|3|^2}{2} = \frac{9}{2} = 4.5$

(a.ii) $x(t) = g(t)\cos(10t) = \left(\frac{3}{2}e^{j30^\circ}e^{j10t} + \frac{3}{2}e^{-j30^\circ}e^{-j10t}\right)\left(\frac{1}{2}\left(e^{j10t} + e^{-j10t}\right)\right)$

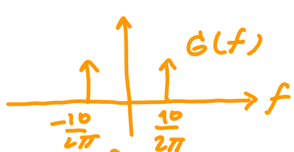
$$= \frac{3}{4}\left(e^{j30^\circ}e^{j20t} + \underbrace{e^{-j30^\circ}e^{0} + e^{j30^\circ}e^{0}}_{=2\cos 30^\circ = \sqrt{3}} + e^{-j30^\circ}e^{-j20t}\right)$$

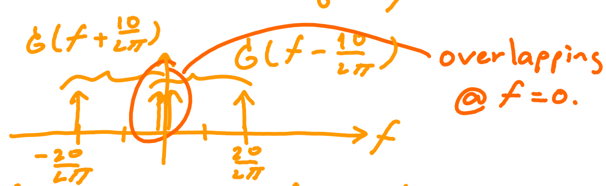
Euler's formula

← Need to combine them first because they have the same frequency.

$$P_x = \left(\frac{3}{4}\right)^2\left(1^2 + (\sqrt{3})^2 + 1^2\right) = \frac{9}{16}(1+3+1) = \frac{45}{16} \approx 2.813$$

Note that although $x(t) = g(t)\cos(2\pi f_0 t)$, we can't use $P_x = \frac{1}{2}P_g$ because $G(f-f_0)$ and $G(f+f_0)$ overlap in the frequency domain.





In general, for $x(t) = a\cos(2\pi f_0 t + \theta)\cos(2\pi f_0 t + \phi)$, applying the product-to-sum formula gives

$$x(t) = \frac{a}{2}\left(\cos\left(2\pi(2f_0)t + \theta + \phi\right) + \cos(\theta - \phi)\right)$$

When $f_0 \neq 0$, the two cosine components do not overlap in the frequency domain. Hence, the power of their sum is the same as the sum of their power.

Therefore, $P_x = \left|\frac{a}{2}\right|^2\left(\frac{1}{2} + \cos^2(\theta - \phi)\right)$.

Here, $a = 3$, $\theta = 0$, $\phi = 30^\circ$.

Therefore, $P_x = \left(\frac{3}{2}\right)^2\left(\frac{1}{2} + \cos^2(30^\circ)\right) = \frac{9}{4}\left(\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2\right) = \frac{9}{4}\left(\frac{1}{2} + \frac{3}{4}\right) = \frac{45}{16}$

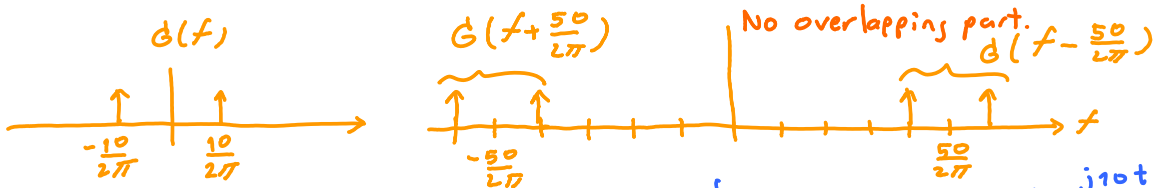
(a.iii) $y(t) = g(t)\cos(50t) = \frac{3}{2}\left(e^{j30^\circ}e^{j10t} + e^{-j30^\circ}e^{-j10t}\right)\frac{1}{2}\left(e^{j50t} + e^{-j50t}\right)$

$$= \frac{3}{4}\left(e^{j30^\circ}e^{j60t} + e^{-j30^\circ}e^{j40t} + e^{j30^\circ}e^{-j40t} + e^{-j30^\circ}e^{-j60t}\right)$$

All of the complex exponential functions have distinct frequencies.

$$P_y = \left(\frac{3}{4}\right)^2\left(1^2 + 1^2 + 1^2 + 1^2\right) = \frac{9}{16} \times 4 = \frac{9}{4} \approx 2.25$$

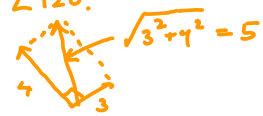
Note that $P_y = \frac{1}{2}P_g$ because $G\left(f - \frac{50}{2\pi}\right)$ and $G\left(f + \frac{50}{2\pi}\right)$ do not overlap.



(b.i) $g(t) = 3\cos(10t + 30^\circ) + 4\cos(10t + 120^\circ) = \mathrm{Re}\left\{\underbrace{(3\angle 30^\circ + 4\angle 120^\circ)}_{\approx 0.5981 + 4.9641j \approx 5\angle 83.13^\circ} e^{j10t}\right\}$

$$= 5\cos(10t + 83.13^\circ)$$

Note that we do not need the phase 83.13° to calculate the average power. Also, we can get the magnitude "5" simply by noticing the 90° difference between $3\angle 30^\circ$ and $4\angle 120^\circ$.



$$P_g = 5^2 \times \frac{1}{2} = \frac{25}{2} = 12.5$$

(b.ii) From part (a.ii), we have

$$P_x = \frac{a^2}{8}\left(1 + 2\cos^2(\theta - \phi)\right) = \frac{5^2}{8}\left(1 + 2\cos^2 83.13^\circ\right) \approx 3.214$$

(b.iii) Note that $G(f)$ is still at $\pm\frac{10}{2\pi}$ as in part (a.iii).

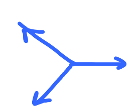
Therefore, $G\left(f - \frac{50}{2\pi}\right)$ and $G\left(f + \frac{50}{2\pi}\right)$ still do not overlap in the freq. domain.

$$P_y = \frac{1}{2}P_g = \frac{25}{4} = 6.25$$

(c.i) Look at the three components of $g(t)$ in their phasor representation.

We have $3\angle 0^\circ + 3\angle 120^\circ + 3\angle 240^\circ = 0$

clear when you draw the three vectors



Therefore, $g(t) = 0$. Hence, $P_g = 0$

(c.ii) $x(t) = 0 \Rightarrow P_x = 0$

(c.iii) $y(t) = 0 \Rightarrow P_y = 0$

Q3 Parseval's Theorem and Energy Calculation

Wednesday, July 18, 2012 9:32 PM

(a) The question itself actually gives us one way to find the total energy :

$$E = \int_{-\infty}^{\infty} |G(f)|^2 \, df.$$

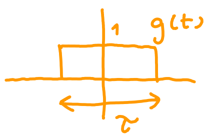
By the Parseval's theorem, we know that this is the same as

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt$$ which is easier to calculate.

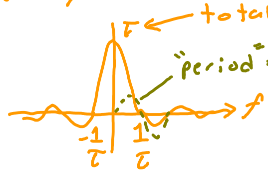
For $g(t) = 1[-1 \le t \le 1]$, the total energy is $\int_{-\infty}^{\infty} \left(1[-1 \le t \le 1]\right)^2 dt = \int_{-1}^{1} 1 \, dt = 2$.

Suppose we want to work in the frequency domain. We will first need to find $G(f)$.

Recall that



$\xrightarrow{\mathcal{F}}$



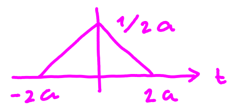
Here, $\tau = 2$. So, $G(f) = 2\,\mathrm{sinc}(2\pi f)$

$$E = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} 2^2 \mathrm{sinc}^2(2\pi f)\, df \leftarrow$$ This is not an easy integral to work with.

$a = 1$

$= 4 \times \frac{1}{2 \times 1} = 2$

In HW2, we've derived one transform pair :



$\xrightarrow{\mathcal{F}} \mathrm{sinc}^2(2\pi a f)$

Therefore, $\int_{-\infty}^{\infty} \mathrm{sinc}^2(2\pi a f)\, df = \frac{1}{2a}$.

$a = \frac{1}{2\pi}$

For quick calculation, it may be useful to remember that $\int_{-\infty}^{\infty} \mathrm{sinc}^2(x)\, dx = \pi$,

$\int_{-\infty}^{\infty} \tau\, \mathrm{sinc}(\pi f \tau)\, df = 1 \Rightarrow \int_{-\infty}^{\infty} \mathrm{sinc}(\pi \tau f)\, df = \frac{1}{\tau}$ and $\int_{-\infty}^{\infty} \mathrm{sinc}(x)\, dx = \pi$.

$\tau = \frac{1}{\pi}$

(b) If you have not found $G(f)$ in part (a), this part require you to do so as the first step.
However, we've already done this as an alternative solution for part (a).
So, we will use that for this part.

The main lobe occupies an interval of frequency from $f_1 = -\frac{1}{\tau} = -\frac{1}{2}$ to $f_2 = +\frac{1}{\tau} = +\frac{1}{2}$.

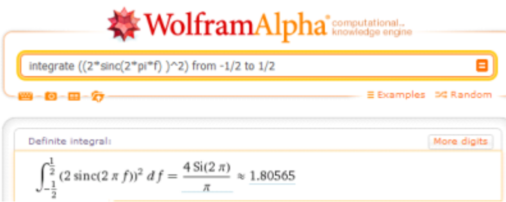
So, the energy contained in the band $B = [f_1, f_2]$ is given by

$$\int_{-1/2}^{1/2} \left(2\,\text{sinc}(2\pi f)\right)^2 df \approx 1.8056$$

MATLAB

or

Wolfram Alpha



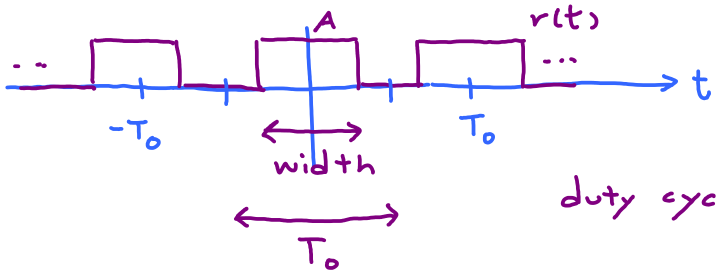
The fraction of energy contained in the main lobe is $\approx \frac{1.8056}{2} \approx 0.9028 = 90.28\%$

the answer from part (a)

(c) Using MATLAB, we can look at the fraction of energy as a function of f_0.
We found that at around $f_0 \approx 5.1$, the fraction begins to exceed 99%.

Q4 Square Wave: Fourier coefficients and duty cycle

Wednesday, October 21, 2015 5:54 PM



$$\text{duty cycle} = \frac{\text{width}}{T_0}$$

(a) In class, we've seen that when the duty cycle is $\frac{1}{n}$, the n^{th} harmonic (along with its multiples) is suppressed. Here, $c_4 = 0$. So, we conclude that the duty cycle is

$$\frac{1}{4} = 25\%.$$

(b) Recall that $c_k = \frac{1}{T_0} \int_{T_0} r(t)\, e^{-j 2\pi k f_0 t}\, dt.$

Therefore, $c_0 = \frac{1}{T_0} \int_{T_0} r(t)\, dt = \langle r(t) \rangle$ ← average.

From the picture, $\langle r(t) \rangle = \frac{\text{width} \times A}{T_0} = (\text{duty cycle}) \times A$.

Therefore, $A = \frac{\langle r(t) \rangle}{\text{duty cycle}}$

We are given that $c_0 = \frac{1}{2}$ and we found, in part (a), that duty cycle $= \frac{1}{4}$.

Therefore, $A = \frac{1/2}{1/4} = 2$.

Q5 Poisson Sum Formula [M2011 Q5]

Monday, August 06, 2012 7:49 PM

(a) and (b) Recall that $\sum_k \delta(t-kT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0}\sum_k \delta(f-kf_0)$ where $f_0 = \frac{1}{T_0}$.

Of course, you may not remember the above fact. However, I asked you to remember one special case which is much easier to remember:

$$\sum_n \delta(t-n) \xrightarrow{\mathcal{F}} \sum_k \delta(f-k)$$

Let this be $y(t)$.

The special case can be turned into the general case via the scaling property:

From $y(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} Y\left(\frac{f}{a}\right)$, we have

$$\sum_n \delta(at-n) = \frac{1}{|a|}\sum_n \delta\left(t-\frac{n}{a}\right)$$

$$\frac{1}{|a|}\sum_k \delta\left(\frac{f}{a}-k\right) = \frac{|a|}{|a|}\sum_k \delta(f-ak)$$

Recall that $\delta(at) = \frac{1}{|a|}\delta(t)$

Therefore

$$\sum_n \delta\left(t-\frac{n}{a}\right) \xrightarrow{\mathcal{F}} |a| \sum_k \delta(f-ak)$$

Let $a = \frac{1}{T_0}$. We then have $\sum_n \delta(t-nT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0}\sum_k \delta\left(f-\frac{k}{T_0}\right)$

Alternatively, one may always go back to the Fourier series formula to obtain such relationship.

In this question, this property is applied to $\sum_\ell \delta(t-\ell T)$ to get

$$\sum_\ell \delta(t-\ell T) \xrightarrow{\mathcal{F}} \frac{1}{T}\sum_\ell \delta\left(f-\frac{\ell}{T}\right)$$

So, by the convolution-in-time rule, we have $x(t) \xrightarrow{\mathcal{F}} G(f) * \frac{1}{T}\sum_\ell \delta\left(f+\left(-\frac{\ell}{T}\right)\right)$

(c) and (d)

The integral under consideration is $\int_{-\infty}^{\infty} \underbrace{e^{j2\pi f t} G(f)}_{\text{call this } b(f)} \delta\left(t-\frac{\ell}{T}\right) df$

By the sifting property of δ-function,

$$\int_{-\infty}^{\infty} b(f)\,\delta\left(f-\frac{\ell}{T}\right) df = b\left(\frac{\ell}{T}\right) = e^{j2\pi\frac{\ell}{T}t}\, G\left(\frac{\ell}{T}\right)$$

Summary: $a = \frac{1}{T}$, $b = -\frac{\ell}{T}$, $c = j2\pi\frac{\ell}{T}t$, $d = \frac{\ell}{T}$