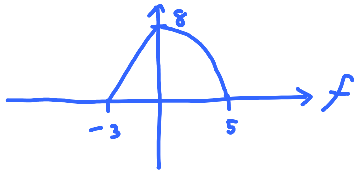
Q1 Time Manipulation and Fourier Transform

Wednesday, September 2, 2015 5:36 PM

In this question, the signal $x(t)$ is not provided directly. Only its Fourier transform $X(f)$ is plotted.

$x(t) \xrightarrow{\mathcal{F}}$



The signal under consideration is $y(t) = x(4-2t) = x(-2(t-2))$.

First, let's consider $g(t) = x(-2t) = x(at)$ where $a = -2$.

By the time scaling property of the Fourier transform,

$$G(f) = \frac{1}{|a|} X\left(\frac{f}{a}\right) \underset{\substack{\uparrow \\ a=-2}}{=} \frac{1}{2} X\left(-\frac{f}{2}\right)$$

Next, note that $y(t) = g(t-2)$.

Recall that time shifting does not change the magnitude of the Fourier transform.

Hence, $|Y(f)| = |G(f)|$.

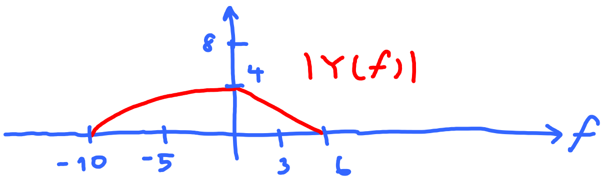
scaled vertically by a factor of $\frac{1}{2}$

no effect because $X(f)$ is already nonnegative.

In conclusion, $|Y(f)| = \left|\frac{1}{2} X\left(-\frac{f}{2}\right)\right| = \frac{1}{2}\left|X\left(-\frac{f}{2}\right)\right|$.

time reversal

expanded horizontally by a factor of 2



Q2 Cosine Pulses

Wednesday, July 18, 2012 3:43 PM

The main purpose of this problem is to see the spectrum of the $\underbrace{\text{cosine pulse}}$.

aka. RF pulse, time-limited sinusoid, finite duration sinusoid.

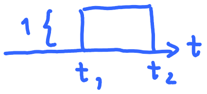
[See C&C p. 59-60]

The pulses under consideration is of the form

$$p(t) = \begin{cases} \cos(2\pi f_0 t), & t_1 \le t \le t_2 \\ 0, & \text{otherwise.} \end{cases}$$

We note that $p(t)$ can be expressed as $p(t) = \cos(2\pi f_0 t) \times r(t)$

where $r(t)$ is the rectangular pulse on the time interval $[t_1, t_2]$.



Writing it in this form makes it clear that we may view $p(t)$ as the modulated signal whose $r(t)$ is the message (or the modulating signal). In which case, we can now apply what we know about modulation:

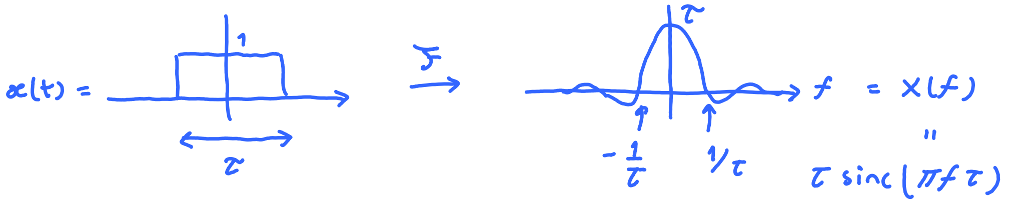
In time domain, $r(t)$ is multiplied by $\cos(2\pi f_0 t)$.

In freq. domain, $R(f)$ is shifted to $\pm f_0$ (and scaled by $\frac{1}{2}$).

In particular, $P(f) = \frac{1}{2}\left(R(f-f_0) + R(f+f_0)\right)$. ★

So, the remaining task is to find $R(f)$.

Recall that



Here, $\tau = t_2 - t_1$.

Moreover, $r(t)$ is the time-shifted version of the $x(t)$ above:

$$r(t) = x\left(t - \frac{t_1+t_2}{2}\right)$$

By the time-shift property,

$$R(f) = \underbrace{e^{-j\cancel{2}\pi f \frac{t_1+t_2}{\cancel{2}}}} X(f) = e^{-j\pi f(t_1+t_2)} (t_2-t_1)\,\text{sinc}\left(\pi f(t_2-t_1)\right) \quad ☆☆$$

Because $\left|\ ^{\smile}\ \right| = 1$, we know that $|R(f)| = |X(f)|$.

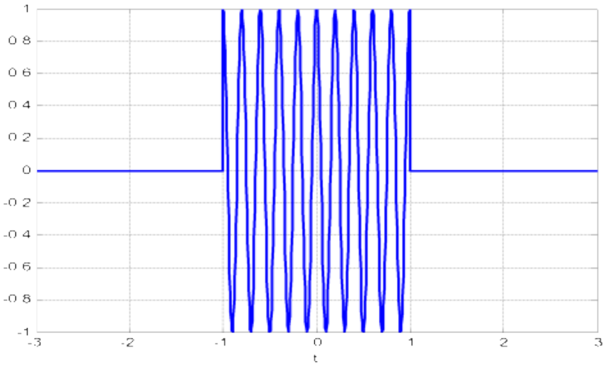
With this, we can get the expression for $P(f)$ from ☆ .

Now, back to the question ...

(a)

(a i)

From MATLAB →



(a.ii) Here, $f_0 = 5$, $t_1 = -1$, and $t_2 = 1$.

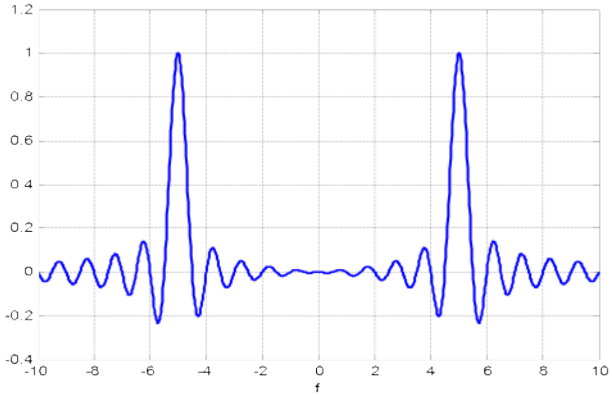
☆☆

Therefore, $R(f) = 2\,\text{sinc}(2\pi f)$

and $P(f) = \text{sinc}\left(2\pi(f-5)\right) + \text{sinc}\left(2\pi(f+5)\right)$ ☆

(a.iii)

From MATLAB →



(b)

(b.i) Here, $f_0 = 5$ (same), $t_1 = 2$, and $t_2 = 4$.

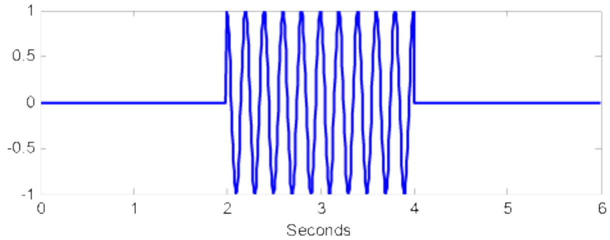
★ ★

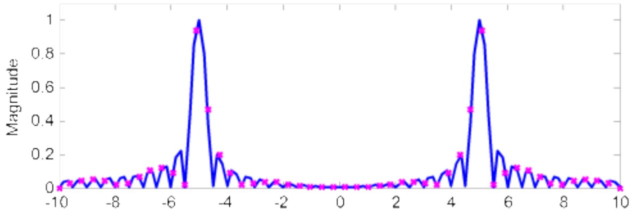
Therefore, $R(f) = e^{-j6\pi f}\, 2\,\mathrm{sinc}(2\pi f)$

and $P(f) = e^{-j6\pi(f-5)}\,\mathrm{sinc}(2\pi(f-5)) + e^{-j6\pi(f+5)}\,\mathrm{sinc}(2\pi(f+5))$ ★

(b.ii)

From MATLAB, →





(b.iii) The 'x' marks above are calculated from the analytical solution in part (b.i). It agrees with what we got in part (b.ii)

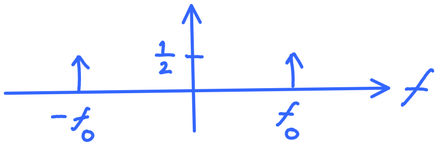
Q3 Tone Modulation

Thursday, July 14, 2011 4:40 PM

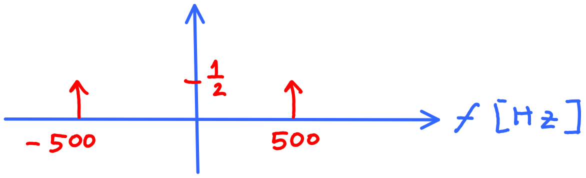
(i.a) $m(t) = \cos 1000\pi t = \cos(2\pi\, 500 t)$

$\hookrightarrow f_0 = 500$ Hz

Recall that the spectrum of $\cos(2\pi f_0 t)$ is given by



So, the spectrum of $m(t)$ is given by



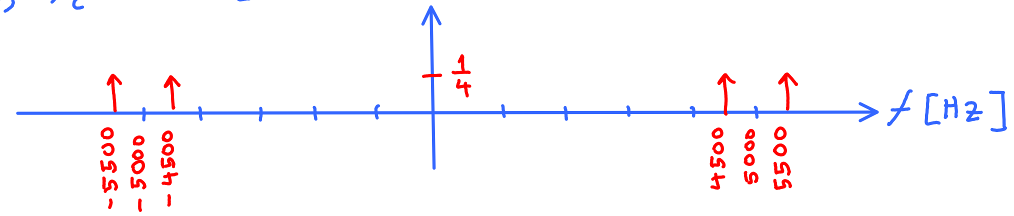
(i.b) Recall that the spectrum of $m(t)\cos(2\pi f_c t)$ is given by

$\frac{1}{2}M(f-f_c) + \frac{1}{2}M(f+f_c)$

↰ shift $M(f)$ to the left by f_c

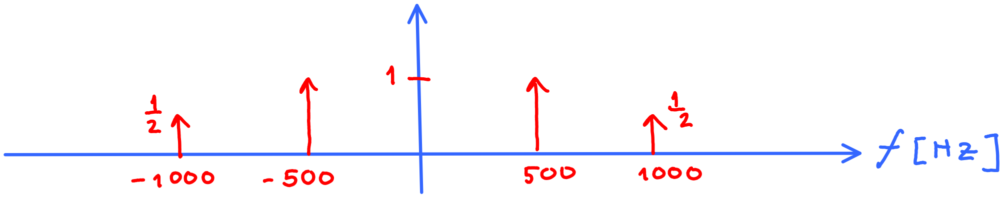
↑ shift $M(f)$ to the right by f_c.

Here, $f_c = 5000$Hz

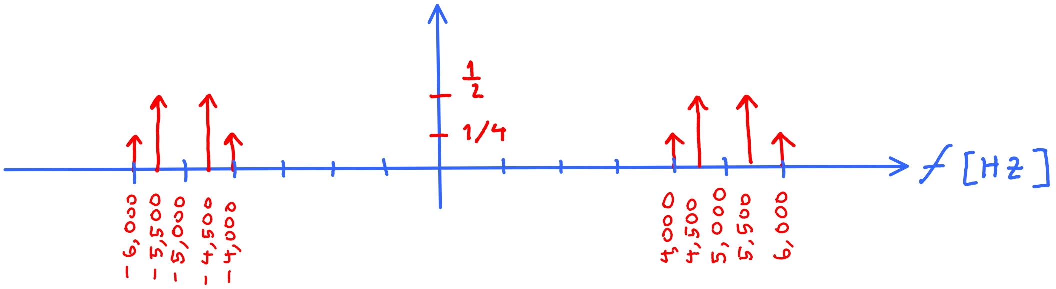


The same explanation applies to parts (ii) and (iii) as well.

(ii.a) $m(t) = 2\cos(1000\pi t) + \cos(2000\pi t) = 2\cos(2\pi \underbrace{500}_{f_1} t) + \cos(2\pi \underbrace{1000}_{f_2} t)$

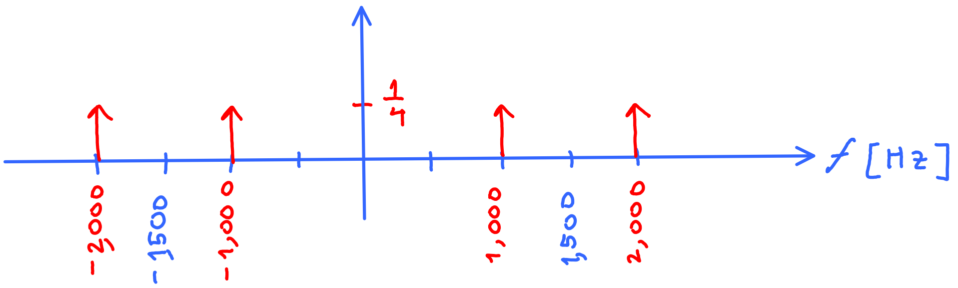


(ii.b)



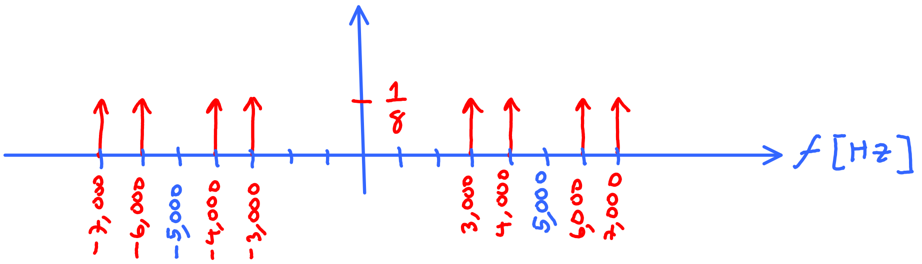
(iii.a) $m(t) = \underbrace{\cos(1000\pi t)}_{\text{part (i.a)}} \times \underbrace{\cos(3000\pi t)}_{= \cos(2\pi(1500)t)}$

from (i.b), multiplication by $\cos(2\pi(1500)t)$ in the time domain is the same as shifting the spectrum content by ± 1500 Hz and vertically scaling by $\frac{1}{2}$.



(iii.b) Here, we multiply again by $\cos(2\pi\, 5{,}000\, t)$.

So, the spectrum from (iii.a) is shifted by ±5,000Hz and scaled vertically by $\frac{1}{2}$.



Q4 Integrations involving sinc function(s)

Thursday, January 22, 2015 6:38 PM

From the Fourier transform review in lecture, we have seen several interesting integration. In particular,

$$\int_{-\infty}^{\infty} G(f)\,df \overset{\star}{=} g(0), \quad \int_{-\infty}^{\infty} g(t)dt \overset{\star\star}{=} G(0), \quad \int_{-\infty}^{\infty} x(t)\,y^*(t)dt \overset{\star\star\star}{=} \int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df.$$

In this question, we use them to evaluate integrals involving sinc functions. Note that direct integration of a sinc function is difficult. However, its Fourier transform is a simple rectangular function which is easy to evaluate or integrate. Therefore, when we see any integral containing sinc function, we first check whether we can perform the integration in another domain.

(a) In this part, we need to evaluate $\int_{-\infty}^{\infty} \operatorname{sinc}(\sqrt{5}x)\,dx$. Note that although variable "x" is used here, we can simply change it to "t" or "f".

Next, in class, we know that $\underbrace{1\left[|t| \le \frac{T_0}{2}\right]}_{g(t)} \xrightarrow{\mathcal{F}} \underbrace{T_0 \operatorname{sinc}(\pi T_0 f)}_{G(f)}$.

So, by $\star$ above, we have $\int_{-\infty}^{\infty} T_0 \operatorname{sinc}(\pi T_0 f)\,df = g(0) = 1$

and hence $\int_{-\infty}^{\infty} \operatorname{sinc}(\pi T_0 f)\,df = \frac{1}{T_0}$.

Here, $\pi T_0 = \sqrt{5}$. Therefore, $T_0 = \frac{\sqrt{5}}{\pi}$.

In which case, $\frac{1}{T_0} = \frac{\pi}{\sqrt{5}}$.

Before we solve parts (b), (c), and (d), we should note that the integrals themselves suggest strongly that we should apply $\star\star\star$ (Parseval's theorem).

(b) The integral is already of the form $\int_{-\infty}^{\infty} X(f)\,Y^*(f)\,df$ where

$$X(f) = e^{-2\pi f \times 2j}\, 2\operatorname{sinc}(2\pi f) = e^{-j2\pi f(2)}\, G(f)$$

$$Y(f) = e^{-2\pi f \times 5j}\, 2\operatorname{sinc}(2\pi f) = e^{-j2\pi f(5)}\, G(f)$$

$$G(f) = 2\operatorname{sinc}(2\pi f)$$

From ★★★ (Parseval's theorem), we know that we can evaluate the integral from $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$. So, we will first find $x(t)$ and $y(t)$.

By the time-shifting property, we know that

$$X(f) = e^{-j2\pi f(2)}\, G(f) \xrightarrow{\mathcal{F}^{-1}} x(t) = g(t-2) \quad \text{and}$$

$$Y(f) = e^{-j2\pi f(5)}\, G(f) \xrightarrow{\mathcal{F}^{-1}} y(t) = g(t-5).$$

Next, from $\underbrace{1\left[|t| \le \frac{T_0}{2}\right]}_{g(t)} \xrightarrow{\mathcal{F}} \underbrace{T_0 \operatorname{sinc}(\pi T_0 f)}_{G(f)}$, by plugging-in $T_0 = 2$, we have

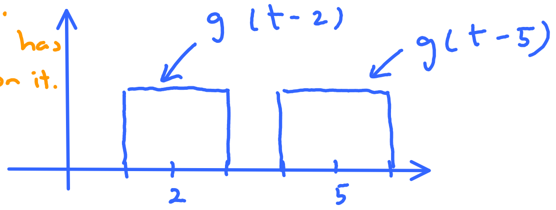
$$G(f) = 2 \operatorname{sinc}(2\pi f) \xrightarrow{\mathcal{F}^{-1}} g(t) = 1\left[|t| \le 1\right].$$

Therefore,

g(t) is real-valued. So, conjugation has no effect on it.

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty} \underbrace{g(t-2)}\ \underbrace{\overset{*}{g}(t-5)}\, dt$$

The non-zero parts of these functions do not overlap. Therefore, their product $\equiv 0$.



$$= \int_{-\infty}^{\infty} 0\, dt = 0$$

(c) We use similar technique to part (b). Here, we need to evaluate

$$\int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df \quad \text{where} \quad X(f) = \operatorname{sinc}(\sqrt{5} f) \quad \text{and} \quad Y(f) = \operatorname{sinc}(\sqrt{7} f).$$

Note that because $Y(f)$ is real-valued, $Y^*(f) = Y(f)$.

From ★★★ (Parseval's theorem), we can replace the integral by $\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$.

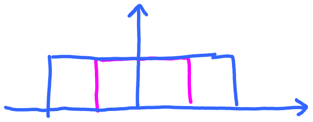
Next, from $1\left[|t| \le \frac{T_0}{2}\right] \xrightarrow{\mathcal{F}} T_0 \operatorname{sinc}(\underbrace{\pi T_0}_{\overset{\shortparallel}{c}} f)$, So, $T_0 = \frac{c}{\pi}$

we have $\frac{\pi}{c}\, 1\left[|t| \le \frac{c}{2\pi}\right] \xrightarrow{\mathcal{F}} \operatorname{sinc}(cf)$.

Therefore,

$$\int_{-\infty}^{\infty} \text{sinc}(c_1 f)\,\text{sinc}^*(c_2 f)\,df = \int_{-\infty}^{\infty} \frac{\pi}{c_1} 1\left[|t| \le \frac{c_1}{2\pi}\right] \frac{\pi}{c_2} 1\left[|t| \le \frac{c_2}{2\pi}\right] dt$$

$$= \frac{\pi^{\cancel{2}}}{c_1 c_2} \times \frac{\min\{c_1, c_2\}}{\cancel{\pi}} = \frac{\pi}{c_1 c_2} \min\{c_1, c_2\}.$$



Here, $c_1 = \sqrt{5}$, $c_2 = \sqrt{7}$.

So, the integral is $\frac{\pi}{\cancel{\sqrt{5}}\sqrt{7}} \cancel{\sqrt{5}} = \frac{\pi}{\sqrt{7}}$

(d) $\text{sinc}(cf) \xrightarrow{\mathcal{F}^{-1}} \frac{\pi}{c} 1\left[|t| \le \frac{c}{2\pi}\right]$

$\downarrow c = \pi$

$\text{sinc}(\pi f) \xrightarrow{\mathcal{F}^{-1}} 1\left[|t| \le \frac{1}{2}\right]$

$\text{sinc}(\pi(f - f_0)) \xrightarrow{\mathcal{F}^{-1}} e^{j2\pi f_0 t} 1\left[|t| \le \frac{1}{2}\right]$ by the frequency-shifting property.

By ★★★ (Parseval's theorem), the integral is the same as

$$\int_{-\infty}^{\infty} e^{j2\pi f_1 t} 1\left[|t| \le \frac{1}{2}\right] e^{-j2\pi f_2 t} 1\left[|t| \le \frac{1}{2}\right] dt = \underbrace{\int_{-\infty}^{\infty} \underbrace{1\left[|t| \le \frac{1}{2}\right]}_{h(t)} e^{-j2\pi(f_2 - f_1)t} dt}$$

Note that the last integral here is exactly the same as the Fourier transform of $h(t)$ evaluated at $f = f_2 - f_1$.

$$= H(f_2 - f_1) = \text{sinc}(\pi(f_2 - f_1))$$

If $f_1 - f_2$ is an integer, then the integral is 0. Here, $f_1 - f_2 = 5 - \frac{7}{2} = \frac{3}{2}$.

So, the integral is $\frac{\sin\left(\frac{3}{2}\pi\right)}{\frac{3}{2}\pi} = \frac{-1}{\frac{3}{2}\pi} = -\frac{2}{3\pi}$.