## Q1 Spectrum via MATLAB

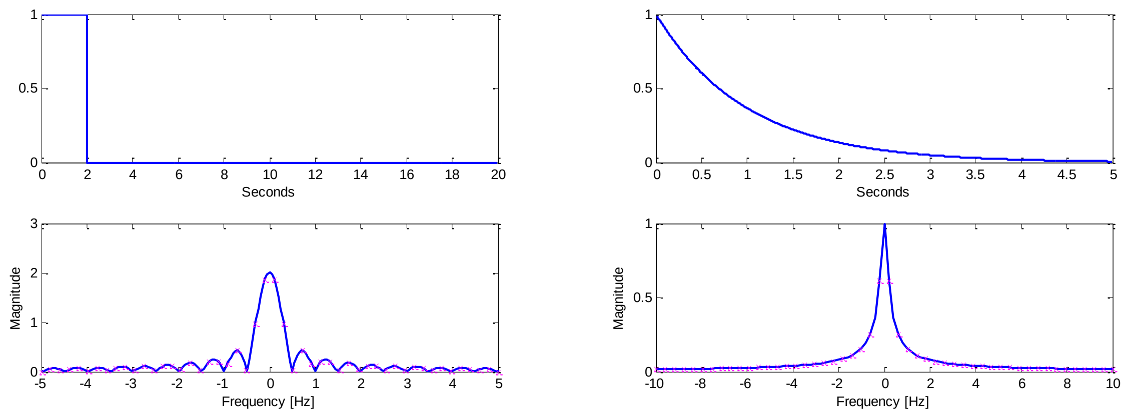
a. You may recall that the Fourier transform of $1\left[|t| \le a\right]$ is given by $2a\,\mathrm{sinc}(2\pi fa)$.

Hence, $1\left[|t| \le 1\right] \xrightleftharpoons{\mathcal{F}} 2\,\mathrm{sinc}(2\pi f)$.

Note that $g(t) = 1[0 \le t \le 2]$ is simply $1\left[|t| \le 1\right]$ time-shifted by 1. As we have discussed in class, time shifting does not change the magnitude of the spectrum. Hence, $|G(f)|$ is the same as the magnitude of the Fourier transform of $1\left[|t| \le 1\right]$. Therefore,

$$|G(f)| = 2\left|\mathrm{sinc}(2\pi f)\right|.$$

In the Figure (i) below, the theoretical expression above is plotted using the “x” marks on top of the provided plot from `specrect.m`. The marks match the theoretical plot.




(i) (ii)

b. See Figure (ii) above.

c.

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} e^{-(1+j2\pi f)t}dt$$

$$= \frac{1}{-(1+j2\pi f)} e^{-(1+j2\pi f)t}\Big|_{t=0}^{\infty} = \frac{1}{1+j2\pi f}$$

$|S(f)|$ is plotted in part (c) using the “x” marks on top of the plots from `plotspec.m`. They are virtually identical.

d. With variable “a” in the m-file set to 1, we have same result.

# Q2 Cosine Pulses

Wednesday, July 18, 2012
3:43 PM

The main purpose of this problem is to see the spectrum of the cosine pulse.

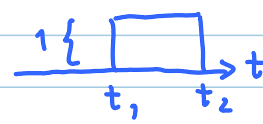
The pulses under consideration is of the form

$$p(t) = \begin{cases} \cos(2\pi f_0 t), & t_1 \le t \le t_2 \\ 0, & \text{otherwise.} \end{cases}$$

We note that $p(t)$ can be expressed as

$$p(t) = \cos(2\pi f_0 t) \times r(t)$$

where $r(t)$ is the rectangular pulse on the time interval $[t_1, t_2]$.




Writing it in this form makes it clear that we may view $p(t)$ as the modulated signal whose $r(t)$ is the message (or the modulating signal).

In which case, we can now apply what we know about modulation:

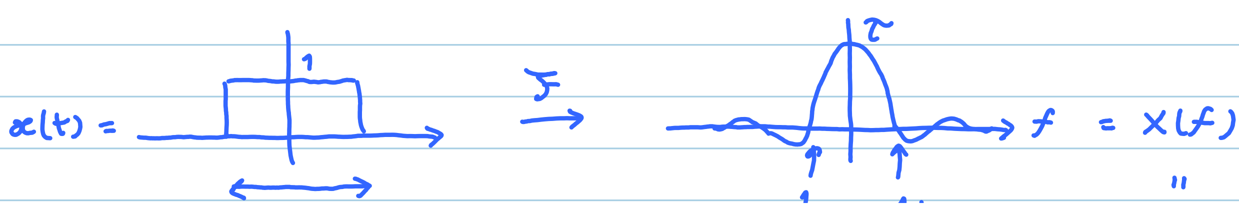
In time domain, $r(t)$ is multiplied by $\cos(2\pi f_0 t)$.

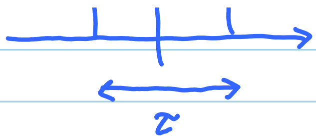
In freq. domain, $R(f)$ is shifted to $\pm f_0$ (and scaled by $\frac{1}{2}$).

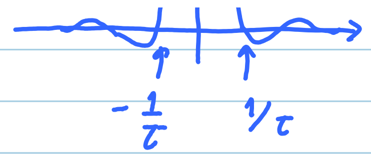
In particular, $P(f) = \frac{1}{2}\left(R(f-f_0) + R(f+f_0)\right)$. ★

So, the remaining task is to find $R(f)$.

Recall that



$x(t) =$ 

$\longrightarrow$ 

$= X(f) = \tau \,\mathrm{sinc}(\pi f \tau)$

Here, $\tau = t_2 - t_1$.

Moreover, $r(t)$ is the time-shifted version of the $x(t)$ above:

$$r(t) = x\left(t - \frac{t_1 + t_2}{2}\right)$$

By the time-shift property,

$$R(f) = e^{-j\cancel{2}\pi f \frac{t_1+t_2}{\cancel{2}}} X(f)$$

Recall that $|R(f)|$ will be the same as $|X(f)|$

$$= e^{-j\pi f (t_1+t_2)} (t_2 - t_1)\,\mathrm{sinc}(\pi f (t_2 - t_1))$$

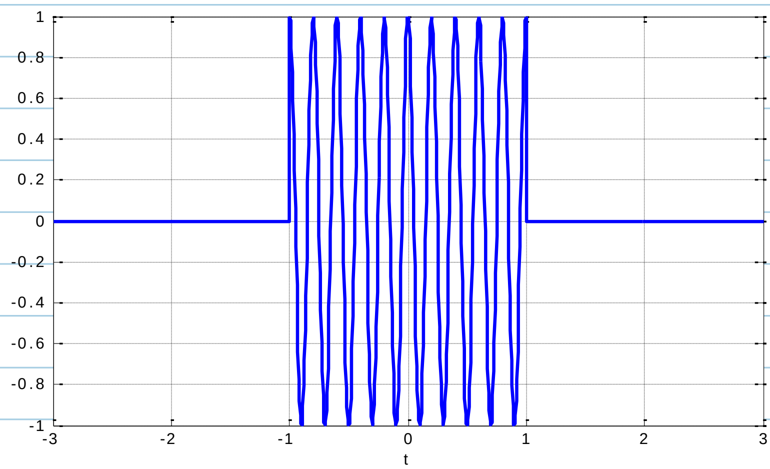
With this, we can get the expression for $P(f)$ from ☆.

Now, back to the question ...

(a)

(a.i)

From MATLAB, $\rightarrow$



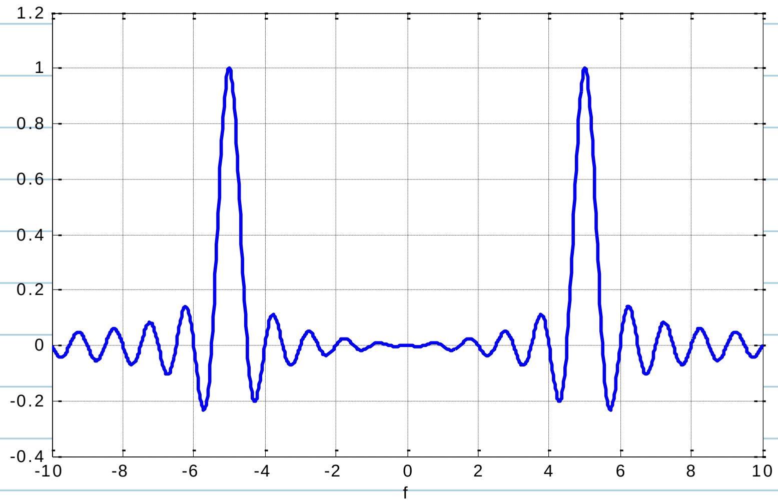

(a.ii) Here, $f_0 = 5$, $t_1 = -1$, and $t_2 = 1$.

Therefore, $R(f) = 2\,\mathrm{sinc}(2\pi f)$

and $P(f) = \mathrm{sinc}(2\pi (f-5)) + \mathrm{sinc}(2\pi (f+5))$

(a.iii)

From $\rightarrow$ MATLAB




(b)

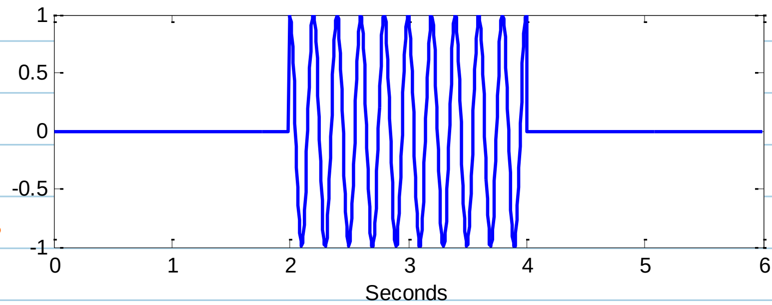
(b.i) Here, $f_0 = 5$ (same), $t_1 = 2$, and $t_2 = 4$.

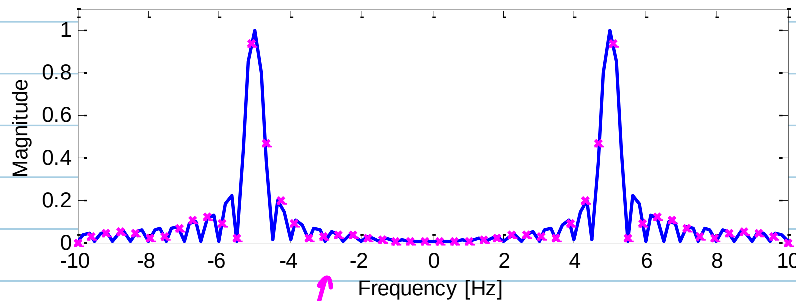
Therefore, $R(f) = e^{-j6\pi f}\, 2\,\mathrm{sinc}(2\pi f)$

and $P(f) = e^{-j6\pi(f-5)}\,\mathrm{sinc}(2\pi(f-5))$
$+\, e^{-j6\pi(f+5)}\,\mathrm{sinc}(2\pi(f+5))$

(b.ii)

From $\rightarrow$ MATLAB




(b.iii) The 'x' marks above are calculated from the analytical solution in part (b.i). It agrees with what we got in part (b.ii).

Q3 Parseval's Theorem and Energy Calculation

Wednesday, July 18, 2012
9:32 PM

(a) The question itself actually gives us one way to find the total energy:

$$E = \int_{-\infty}^{\infty} |G(f)|^2 \, df.$$

By the Parseval's theorem, we know that this is the same as

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt$$

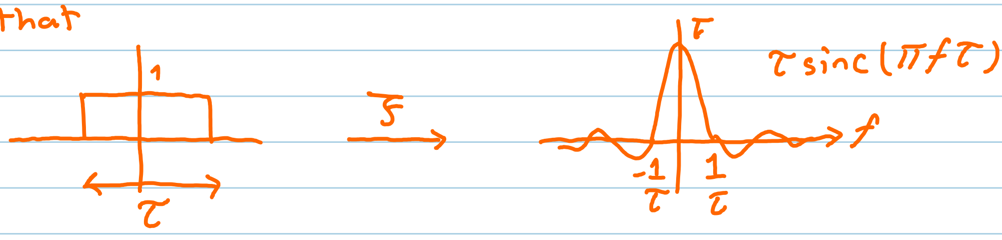
which is much easier to calculate.

For $g(t) = 1[-1 \le t \le 1]$, the total energy is $\displaystyle\int_{-\infty}^{\infty} \left(1[-1 \le t \le 1]\right)^2 dt$

$$= \int_{-1}^{1} 1 \, dt = 2.$$

Alternatively, we can work directly with the integration in the frequency domain. To do this, we will first need to find $G(f)$.

Recall that




Here, $\tau = 2$. So, $G(f) = 2\,\mathrm{sinc}(2\pi f)$ and

$$E = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} 2^2 \mathrm{sinc}^2(2\pi f) \, df$$

$\mu = 2\pi f$

$df = \frac{1}{2\pi} d\mu$

$$= 4 \int_{-\infty} \text{sinc}^2(\mu) \frac{1}{2\pi} d\mu = \frac{2}{\pi} \underbrace{\int_{-\infty} \text{sinc}^2(\mu)\, d\mu}_{\text{we've shown that this is } \pi.}$$

$= 2$

↑ same as the energy found in the time domain (but the integration is considerably more difficult).

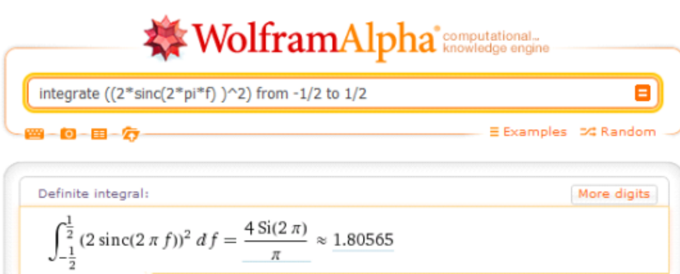
(b) If you have not found $G(f)$ in part (a), this part require you to do so as the first step. However, we've already done this as an alternative solution for part (a). So, we will use that for this part.

The main lope occupies an interval of frequency from $f_1 = -\frac{1}{\tau} = -\frac{1}{2}$ to $f_2 = +\frac{1}{\tau} = +\frac{1}{2}$.

So, the energy contained in the band $B = [f_1, f_2]$ is given by

$$\int_{-1/2}^{1/2} \left(2\,\text{sinc}(2\pi f)\right)^2 df \approx 1.8056$$

↑ MATLAB or wolfram Alpha




The fraction of energy contained in the main lope is

$$\approx \frac{1.8056}{2} \approx 0.9028 = 90.28\%$$

↑ the answer from part (a)

(c) Using MATLAB, we can look at the fraction of energy as a function of $f_0$.

We found that at around $f_0 \approx 5.1$, the fraction begins to exceed 99%.

## Q4 Linear System

Wednesday, July 18, 2012
4:57 PM

One requirement for a linear system is that

"proportional changes in the input give the same proportional changes in the output."

In particular, if $x = 1$ corresponds to $y = 12$,

then $x = 1 \times 2$ should correspond to $y = 12 \times 2 = 24$.

(Doubling the input cause the output to double.)

In our case, we have $y = 2x + 10$.

So, if $x = 1$, $y = 2 \times 1 + 10 = 12$.

For linear system, when $x = 2$, we expect $y$ to be 24.

However, by its definition, when $x = 2$, our system gives

$$y = 2 \times 2 + 10 = 14 \neq 24.$$

Therefore, the system is not linear.

# Q5 Time Delay

Wednesday, July 18, 2012
5:17 PM

$$x(t) = 10\cos(2\pi f_c t)$$

$$y(t) = 10\cos(2\pi f_c t - \theta) = 10\cos\Big(2\pi f_c \big(t - \underbrace{\tfrac{\theta}{2\pi f_c}}_{\text{delay}}\big)\Big)$$

The delay is caused by the propagation time of the signal.

Recall that the amount of time delay can be calculated from

$$\text{delay} = \frac{\text{distance}}{c} \quad \leftarrow \text{speed of light.}$$

Therefore, "one possible" distance value is

$$\text{distance} = c \times \text{delay} = c \times \frac{\theta}{2\pi f_c} = \lambda_c \frac{\theta}{2\pi}$$

wavelength of the carrier

$$= \cancel{3}\times 10^8 \times \frac{\cancel{\pi}/\cancel{6}\ 2}{2\cancel{\pi} \times 7\times 10^6} = \frac{100}{28} \approx 3.57 \text{ m}$$

The calculation above gives only one possible distance value because the cosine is periodic.

In particular,

$$\cos(2\pi f_c t - \theta) = \cos(2\pi f_c t - \theta + 2\pi k) \quad \text{for any integer } k.$$

So, what we should do is to consider

$$\cos(2\pi f_c t - \theta + 2\pi k) = \cos\Big(2\pi f_c \big(t - \frac{\theta}{2\pi f_c} + \frac{k}{f_c}\big)\Big)$$

In which case, the amount of time delay could be

$$\frac{\theta}{2\pi f_c} - \frac{k}{f_c}$$ for any integer $k$.

The corresponding possible values of distance are

$$d = \frac{c}{f_c}\underbrace{\left(\frac{\theta}{2\pi} - k\right)} = \lambda_c\left(\frac{\theta}{2\pi} - k\right)$$

wavelength of the carrier.

The distance is a positive quantity.

So, we need $k < \frac{\theta}{2\pi} = \frac{\cancel{\pi}/6}{2\cancel{\pi}} = \frac{1}{12}$.

In other words, $k$ can be $0, -1, -2, -3, \ldots$

(i)

The value of $k$ which corresponds to the minimum value of distance is $k=0$. The minimum distance is

$$d = \frac{c}{f_c}\frac{\theta}{2\pi} = 3.57 \text{ m}.$$

(ii) Other possible values of the distance are

$$d = \frac{c}{f_c}\left(\frac{\theta}{2\pi} - k\right) \quad \text{for } k = -1, -2, -3, \ldots$$

$$= \frac{c}{f_c}\left(\frac{\theta}{2\pi} + n\right) \quad \text{where } n = 1, 2, 3, \ldots$$

$$= 3.57 + 42.86n \quad \text{where } n = 1, 2, 3, \ldots$$

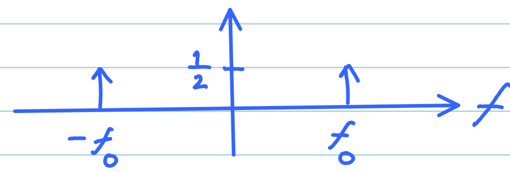
# Q6 Tone Modulation

Thursday, July 14, 2011
4:40 PM

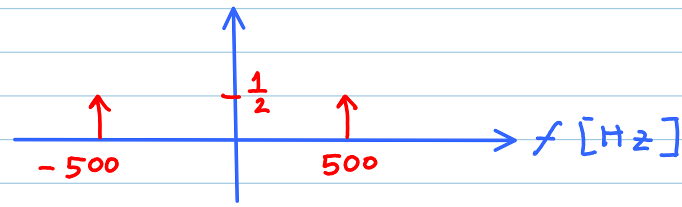
(i.a) $m(t) = \cos 1000\pi t = \cos(2\pi\, 500\, t)$

$\hookrightarrow f_0 = 500$ Hz

Recall that the spectrum of $\cos(2\pi f_0 t)$ is given by




So, the spectrum of $m(t)$ is given by



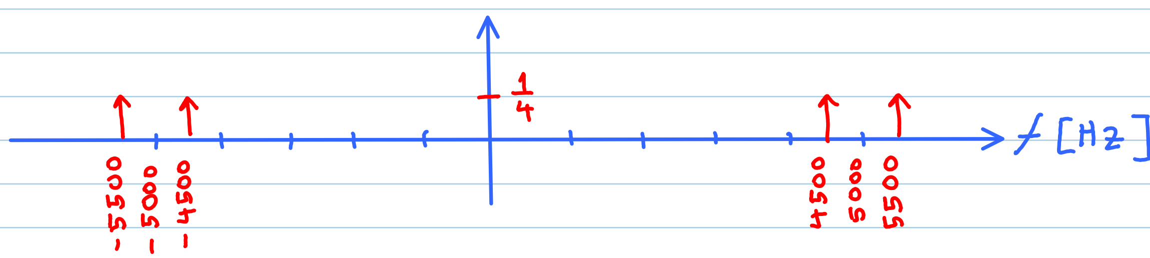

(i.b) Recall that the spectrum of $m(t)\cos(2\pi f_c t)$ is given by

$\frac{1}{2}M(f-f_c) + \frac{1}{2}M(f+f_c)$

shift $M(f)$ to the left by $f_c$

shift $M(f)$ to the right by $f_c$.

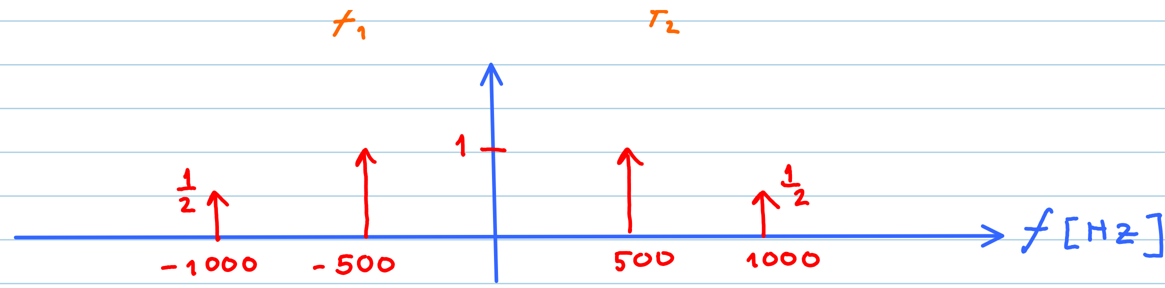
Here, $f_c = 5000$Hz



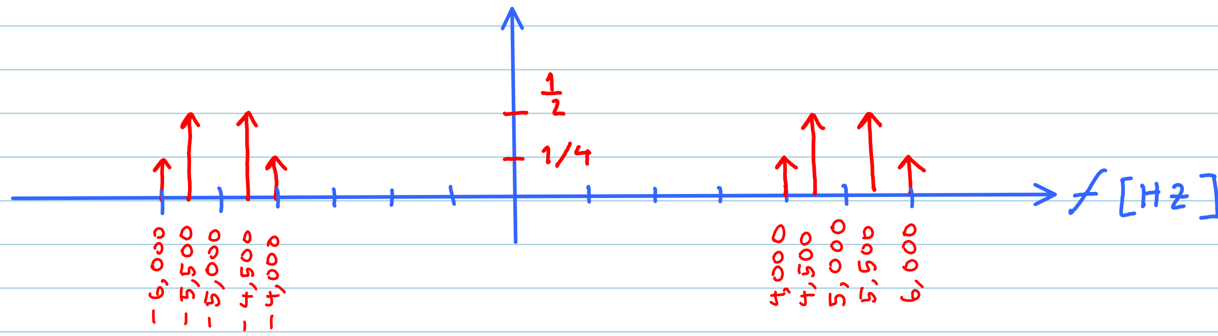

Part (ii) uses the same explanation as part (i).

(ii.a) $m(t) = 2\cos(1000\pi t) + \cos(2000\pi t)$

$= 2\cos(2\pi \underbrace{500}_{f_1} t) + \cos(2\pi \underbrace{1000}_{f_2} t)$

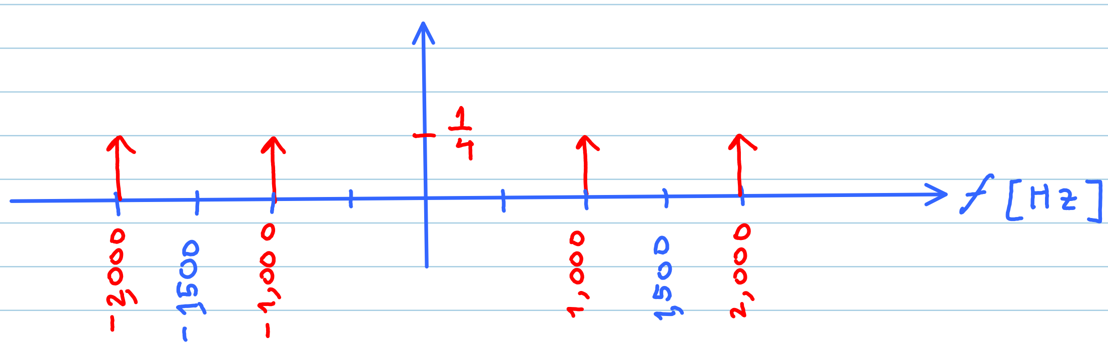



(ii.b)



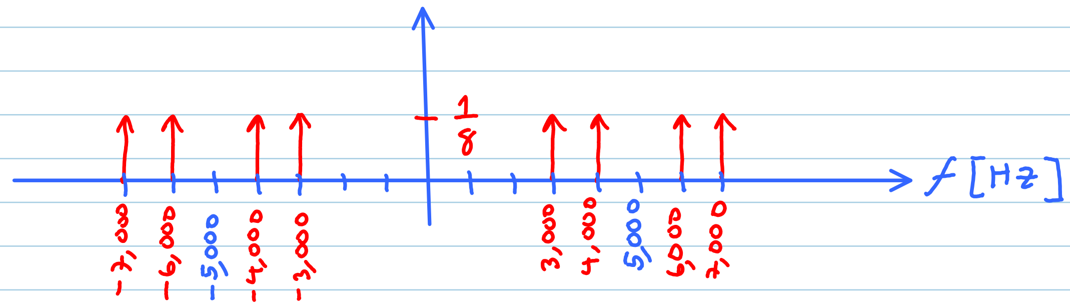

(iii.a) $m(t) = \underbrace{\cos(1000\pi t)}_{\text{part (i.a)}} \times \underbrace{\cos(3000\pi t)}_{= \cos(2\pi(1500)t)}$

from (i.b), multiplication by $\cos(2\pi(1500)t)$ in the time domain is the same as shifting the spectrum content by $\pm 1500$ Hz and vertically scaling by $\frac{1}{2}$.




(iii.b) Here, we multiply again by $\cos(2\pi\, 5{,}000\, t)$.

So, the spectrum from (iii.a) is shifted by $\pm 5{,}000$ Hz and scaled vertically by $\frac{1}{2}$.



1/8
f [Hz]
-7,000
-6,000
-5,000
-4,000
-3,000
3,000
4,000
5,000
6,000
7,000

Q7 Square MODEM

Thursday, July 14, 2011

2:22 PM

(a) $x(t) = A_c m(t) \xrightarrow{\mathcal{F}} X(f) = A_c M(f)$

So, $X(f)$ is also bandlimited to $B$.

$u(t) = x(t) + \sqrt{2}\cos(\omega_c t)$ $\quad \omega_c = 2\pi f_c$

$v(t) = u^2(t) = \left(x(t) + \sqrt{2}\cos(\omega_c t)\right)^2$

$= x^2(t) + 2\sqrt{2}\,x(t)\cos(\omega_c t) + \underbrace{2\cos^2(\omega_c t)}_{1+\cos(2\omega_c t)}$

$= \underbrace{\left(1 + x^2(t)\right)}_{\xrightarrow{BPF} 0} + 2\sqrt{2}\,x(t)\cos\omega_c t + \underbrace{\cos(2\omega_c t)}_{\xrightarrow{BPF} 0}$

Note1: $x^2(t) \xrightarrow{\mathcal{F}} X(f) * X(f)$

So, $x^2(t)$ is bandlimited to $2B$

Because $f_c \gg B$, the spectrum of $x^2(t)$ will not be in the passband of the BPF which centers around $f_c$.

Note 2: The term $\cos(2\omega_c t)$ is at frequency $2 \times f_c$ which again is outside the passband.

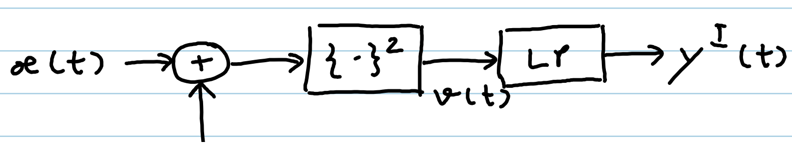
$y(t) = BPF\{v(t)\}$

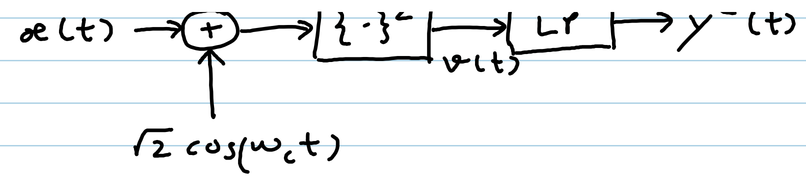
$= 2\sqrt{2}\,x(t)\cos\omega_c t$

$= 2\sqrt{2}\,A_c\, m(t)\cos\omega_c t$

(b) Assume

$x(t) = A_c m(t)\sqrt{2}\cos(\omega_c t)$






From the above figure,

$$v(t) = \left(x(t) + \sqrt{2}\cos(\omega_c t)\right)^2$$

$$= 2\cos^2(\omega_c t)\left(A_c m(t) + 1\right)^2$$

$$= 1 + \cos(2\omega_c t)\underbrace{\left(A_c^2 \underbrace{m^2(t)}_{\substack{\text{spectrum}\\ \text{is from}\\ [-2B,\,2B]}} + 1 + 2A_c \underbrace{m(t)}_{\substack{\text{spectrum}\\ \text{is from}\\ [-B,\,B]}}\right)}_{g(t)}$$

$$= g(t) + \underbrace{g(t)\cos(2\omega_c t)}_{\xrightarrow{\text{LPF}}\ 0}$$

Note 1: We know that $g(t)$ is band limited to $[-2B, 2B]$ because all of its terms are band limited to $[-2B, 2B]$. So, only some parts of it will pass through the LPF.

Note 2: $g(t)\cos(2\omega_c t)$ is centered @ $2f_c$ and therefore will not pass thought the LPF.

$$y^I(t) = \text{LPF}\{v(t)\}$$

$$= \text{LPF}\{g(t)\}$$

$$= 1 + 2A_c m(t) + \text{LPF}\{A_c^2 m^2(t)\}$$

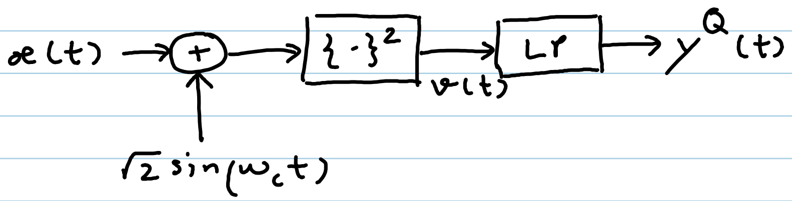
This term has spectrum beyond IW
So, only a portion of it will pass through the LPF.

$y^I(t)$ is not proportional to $m(t)$.

Hence, this block diagram does not work as a demodulator.

(c) Assume

$x(t) = A_c m(t)\sqrt{2}\cos(\omega_c t)$ as in part (b).




We then have

$$v(t) = \left(x(t) + \sqrt{2}\sin(\omega_c t)\right)^2$$
$$= 2\left(A_c m(t)\cos(\omega_c t) + \sin(\omega_c t)\right)^2$$
$$= 2\left(A_c^2 m^2(t)\cos^2(\omega_c t) + A_c m(t)\cos(\omega_c t)\sin(\omega_c t) + \sin^2(\omega_c t)\right)$$
$$= 2\left(A_c^2 m^2(t)\cos^2(\omega_c t) + \sin^2(\omega_c t)\right) + A_c m(t)\sin(2\omega_c t)$$
$$= 2\left(\left(A_c^2 m^2(t) - 1\right)\cos^2(\omega_c t) + 1\right) + A_c m(t)\sin(2\omega_c t)$$
$$= 2 + \left(A_c^2 m^2(t) - 1\right)\Big(1 + \underbrace{\cos(2\omega_c t)}_{\text{LPF} \to 0}\Big) + A_c m(t)\underbrace{\sin(2\omega_c t)}_{\text{LPF} \to 0}$$

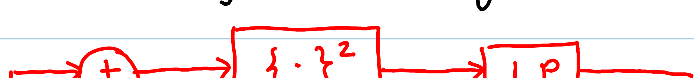
$$y^Q(t) = 2 + \mathrm{LPF}\left\{A_c^2 m^2(t)\right\} - 1$$
$$= \mathrm{LPF}\left\{A_c^2 m^2(t)\right\} + 1$$

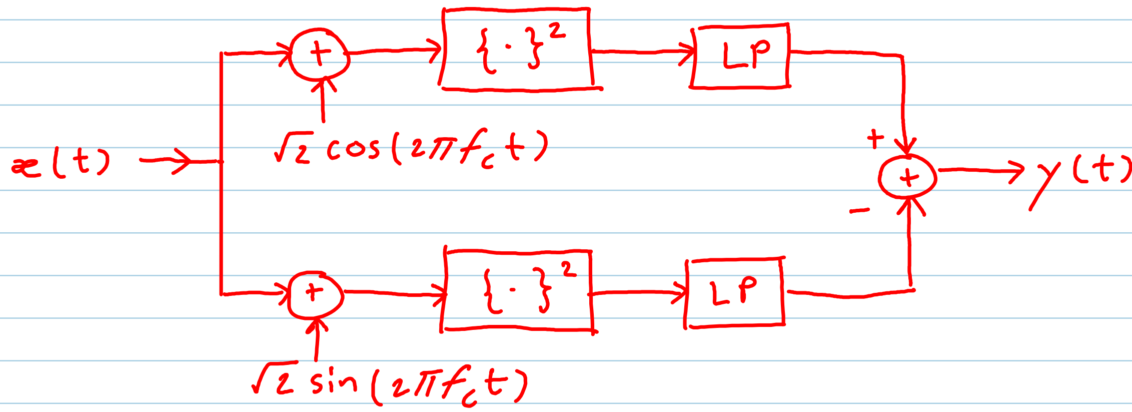
(d) Observe that

$$\underbrace{y^I(t)}_{\text{from (b)}} - \underbrace{y^Q(t)}_{\text{from (c)}} = 2A_c m(t)$$

which is the desired output of a successful DSB-SC demodulator.

Hence, the following block diagram would work:





x(t)
√2 cos(2πf_c t)
{·}^2
LP
+
−
y(t)
√2 sin(2πf_c t)

# Q8 Cube Modulator

Thursday, July 14, 2011
2:11 PM

(a) $y(t) = \left(m(t) + \sqrt{2}\cos(2\pi f_0 t)\right)^3$

$$= m^3(t) + 3m^2(t)\sqrt{2}\cos\omega_0 t + \underbrace{3m(t)\,2\cos^2\omega_0 t}_{\substack{= 3m(t)(1+\cos 2\omega_0 t) \\ = 3m(t) + 3m(t)\cos(2\omega_0 t)}} + \underbrace{(\sqrt{2})^3\cos^3(\omega_0 t)}_{= \frac{3}{\sqrt{2}}\cos(\omega_0 t) + \frac{1}{\sqrt{2}}\cos(3\omega_0 t)}$$

$$\left\{\begin{aligned} 2\cos^2(\theta) &= 1 + \cos(2\theta) \\ 2\cos^3(\theta) &= \cos\theta + \cos\theta\cos 2\theta \\ &= \cos\theta + \tfrac{1}{2}\cos\theta + \tfrac{1}{2}\cos 3\theta \\ &= \tfrac{3}{2}\cos\theta + \tfrac{1}{2}\cos 3\theta \end{aligned}\right\}$$

We want $z(t) = m(t)\sqrt{2}\cos(\omega_c t)$.

We see that the only term in $y(t)$ that has the form

constant × m × cos( )

is $3m(t)\cos(2\omega_0 t)$.

Therefore, we will center the passband to cover this part and adjust the gain to make the output the same as $z(t)$.

In particular,

We need to make $2f_0 = f_c$. So, $f_0 = f_c/2$.

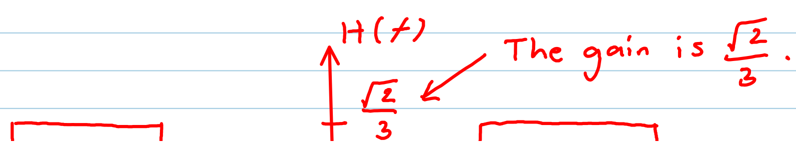
Let $H_{BP}(f) = \begin{cases} c, & |f - f_c| \le B \\ c, & |f + f_c| \le B \\ 0, & \text{otherwise} \end{cases}$

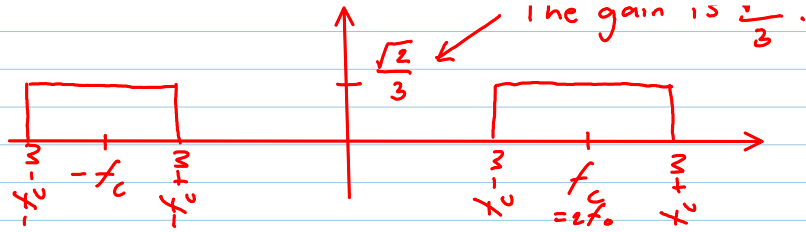
Then, $z(t) = \underbrace{c \times 3}\, m(t)\cos(2\omega_0 t)$

↓

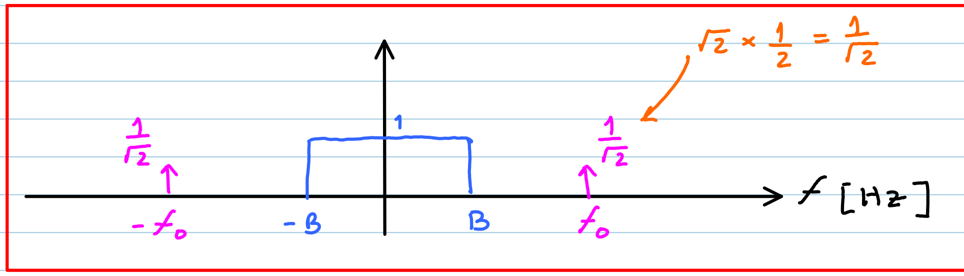
we need $c \times 3 = \sqrt{2} \Rightarrow c = \frac{\sqrt{2}}{3}$

The plot of $H(f)$ is given below:






(b.i) $x(t) = m(t) + \sqrt{2}\cos(2\pi f_0 t)$




(b.ii)

From (a), we have

$$y(t) = m^3(t) + 3\sqrt{2}\, m^2(t)\cos(\omega_0 t) + 3m(t)\cos(2\omega_0 t) + \frac{1}{\sqrt{2}}\cos(3\omega_0 t)$$

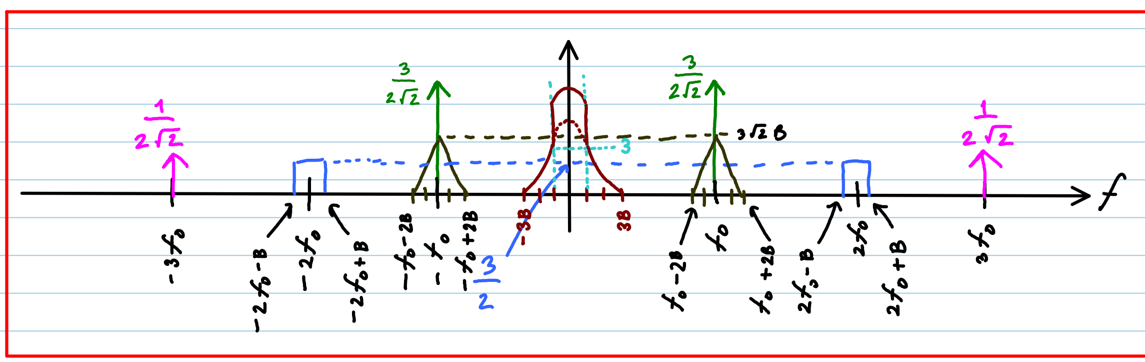
$+3m(t)$

$+\frac{3}{\sqrt{2}}\cos(\omega_0 t)$

Without trying to make and accurate plot for $m^3(t)$, we know that it is bandlimited to $3B$.

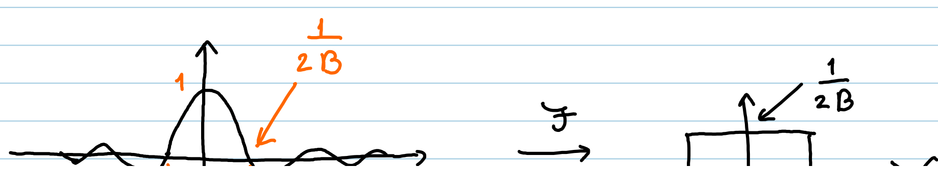
If you want to know the shape of $M(f) * M(f) * M(f)$, you can try plotting it in MATLAB using this code:

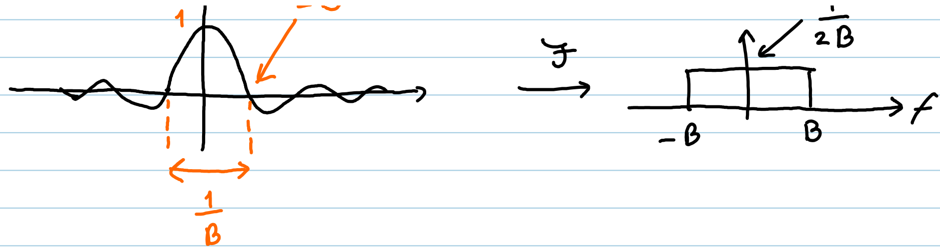
```
u = ones(1,10);
u2 = conv(u,u);
u3 = conv(u2,u);
plot(u3)
```



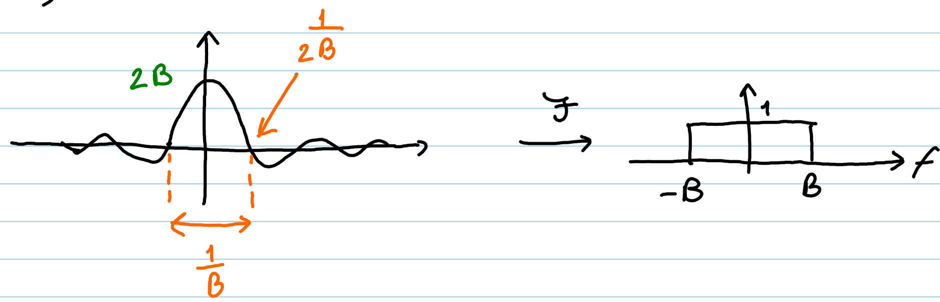

(c) $z(t) = m(t)\sqrt{2}\cos(2\pi f_c t)$

We know that






So,



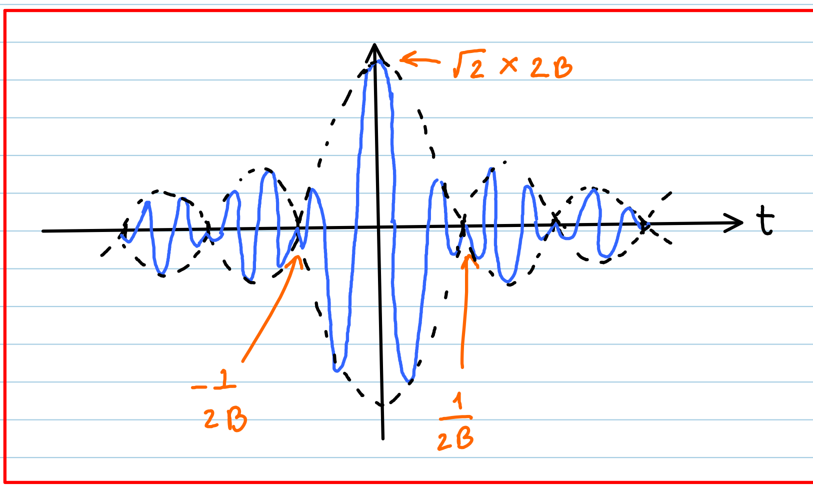

$z(t)$ is the above sinc. function multiplied by $\sqrt{2}\cos(2\pi f_c t)$.

Because $f_c \gg B$, we know that

$$\frac{1}{B} \gg \frac{1}{f_c}$$

↑ period of cos.

So, the sinc function becomes the envelope of the cosine carrier.



(a) First, we use the product-to-sum formula

$$\cos(A)\cos(B) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

to expand $\cos^3 x$ into sum of weighted $\cos(kx)$.

$$\cos^2 x = \cos x \cos x = \frac{1}{2}\left(\cos(2x) + \cos(0)\right) = \frac{1}{2}\left(\cos 2x + 1\right)$$

$$\cos^3 x = \cos x \cos^2 x = \cos x \left(\frac{1}{2}\left(\cos 2x + 1\right)\right)$$

$$= \frac{1}{2}\left(\underbrace{\cos x \cos 2x}_{=\frac{1}{2}\left(\cos(3x) + \cos x\right)} + \cos x\right) = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

Plugging in $x = \omega_c t = 2\pi f_c t$, we get

$$\cos^3 \omega_c t = \frac{1}{4}\cos(3\omega_c t) + \frac{3}{4}\cos(\omega_c t).$$

At point (c), we want $k\, m(t) \cos \omega_c t$

At point (b), we have

$$m(t)\cos^3\omega_c t = \underbrace{\frac{1}{4} m(t) \cos(3\omega_c t)}_{\text{don't want this part}} + \underbrace{\frac{3}{4} m(t) \cos(\omega_c t)}_{\text{want this part}}.$$

Any bandpass filter centered at $\pm\omega_c$ will work.

In addition, the passband of this filter must be larger than $2B$.

Note that if the gain of the BPF is 1, then $k = \frac{3}{4}$.

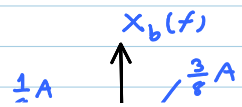
(b)

(b.1) Let $x_b(t)$ be the signal at point (b).

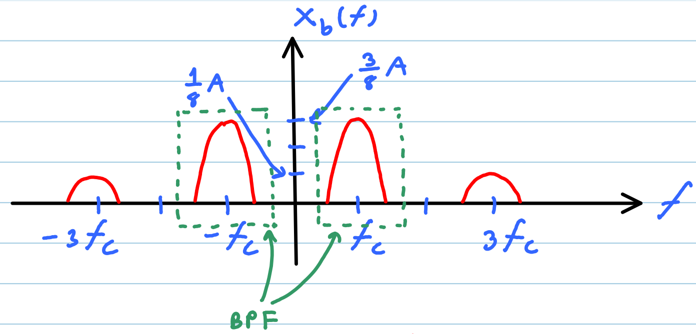
Then $x_b(t) = m(t)\cos^3\omega_c t = \frac{1}{4} m(t)\cos(3\omega_c t) + \frac{3}{4} m(t) \cos(\omega_c t)$

$$\xrightarrow{\mathcal{F}} \frac{1}{8} M(f - 3f_c) + \frac{1}{8} M(f + 3f_c) + \frac{3}{8} M(f - f_c) + \frac{3}{8} M(f + f_c)$$

where $f_c = \omega_c / 2\pi$.



where $f_c = \omega_c / 2\pi$.




The frequency bands occupied are $[-3f_c - B, -3f_c + B]$,
$[-f_c - B, -f_c + B]$,
$[f_c - B, f_c + B]$, and
$[3f_c - B, 3f_c + B]$.

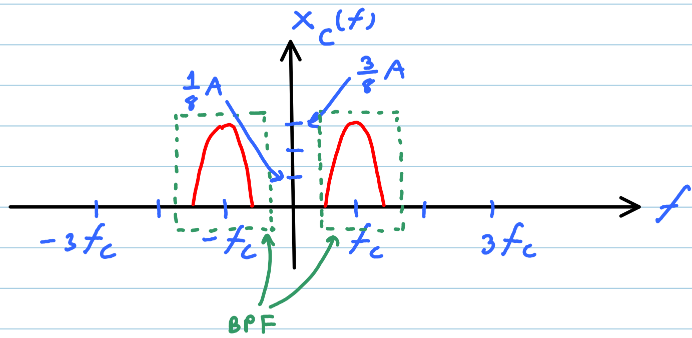
(b.2) Let $x_c(t)$ be the signal at point c.

We will assume that the gain of the BPF is 1.

(In general, if gain $= g$, then $k = \frac{3}{4} g$)

In which case, $x_c(t) = \frac{3}{4} m(t) \cos \omega_c t$

and $X_c(f) = \frac{3}{8} M(f - f_c) + \frac{3}{8} M(f + f_c)$




The frequency bands occupied are $[-f_c - B, -f_c + B]$
and $[f_c - B, f_c + B]$

(c) To avoid overlapping of spectra at point b,

we must have

$f_c - B > 0$, and

$$f_c + B < 3f_c - B.$$

Both conditions require $f_c > B$.

Hence, the mininum usable value of $f_c$ is B.

(d) Recall (from part (a)) that $\cos^2\omega_c t = \frac{1}{2} + \frac{1}{2}\cos(2\omega_c t)$.

There is no component around $f_c$.

Hence, this system would not give the desired output.

(e) For this question, if you have the time to derive or look up general formula for the expansion of $\cos^n \omega_c t$ into a weighted sum of $\cos k\omega_c t$, then you can skip most of the explanation below.

First, observe that $\cos^n(\omega_c t)$ is a periodic funtion with "period" $T_c = 1/f_c = 2\pi/f_c$.

This is the period of $\cos(\omega_c t)$.
Any function of $\cos \omega_c t$ will automatically repeat itself every $T_c$.

It is also an even function.
So, its Fourier series is given by

$$c_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_c t).$$

The "even" property kills the $b_k \sin k\omega_c t$ parts.

So,

$$m(t)\cos^n(\omega_c t) = c_0 m(t) + \sum_{k=1}^{\infty} a_k m(t) \cos k\omega_c t$$

We know that the Fourier transform of $a_k m(t)\cos(k\omega_c t)$ is just scaled replicas of $M(f)$ centered at $\pm k f_c$.

So, the Fourier trans from of $m(t)\cos^n(\omega_c t)$ is simply $M(f)$ scaled and replicated at $0, \pm f_c, \pm 2f_c, \pm 3f_c, \ldots$

We want the component at $f_c$ and hence the problem is reduced to figuring out whether there is a replica of $M(f)$ at $f_c$.

abbreviated iff

reduced to figuring out whether there is a replica of $r(t)$

at $f_c$.

abbreviated iff

In other words, the scheme works if and only if $a_1 \neq 0$.

From our notes, we have

$$a_k = \frac{2}{T_c} \int_{T_c} \underbrace{\cos^n(\omega_c t)}_{} \cos(k\omega_c t)\, dt.$$

you don't need to use this factor because we only want to determine whether $a_1 > 0$

This is our $r(t)$ in the notes. Here, $r(t) = \cos^n(\omega_c t)$

So,

$$a_1 \neq 0 \quad \text{iff} \quad \int_{T_c} \cos^{n+1} \omega_c t \, dt \neq 0.$$

When $n$ is odd, $n+1$ is even

$\cos^{n+1} \omega_c t$ is strictly positive almost everywhere.

Therefore, the integral is strictly positive, and $a_1$ is also strictly positive.

Before we consider the case when $n$ is even, first note that we can write

$$\int_{T_c} \cos^{n+1} \omega_c t \, dt = \int_0^{T_c} \cos^{n+1} \omega_c t \, dt$$

$$= \int_0^{T_c/2} \cos^{n+1} \omega_c t \, dt + \underbrace{\int_{T_c/2}^{T_c} \cos^{n+1} \omega_c t \, dt}_{}$$

Let $\tau = t - \frac{T_c}{2}$

$$= \int_0^{T_c/2} \cos^{n+1}\left(\omega_c\left(\tau + \frac{T_c}{2}\right)\right) d\tau$$

$$= \int^{T_c/2} \cos^{n+1}(\omega_c \tau + \pi)\, d\tau$$

$\cos(x+\pi) = -\cos x$

$$= (-1)^{n+1}\int_0^{T_c/2} \cos^{n+1}(\omega_c \tau)\, d\tau$$

So, $\int_{T_c} \cos^{n+1}\omega_c t\, dt = A + (-1)^{n+1} A$ where $A = \int_0^{T_c/2} \cos^{n+1}\omega_c t\, dt$

When $n$ is even, $(-1)^{n+1} = -1$ and

$$\int_{T_c} \cos^{n+1}\omega_c t\, dt = A + (-1)A = A - A = 0.$$

Therefore, $a_1 = 0$.

In conclusion, the identity for $\cos^n \omega_c t$ contains a term $\cos\omega_c t$ when $n$ is odd. This is not true when $n$ is even.

Therefore, the system works if and only if $n$ $(\geq 2)$ is odd.