Q1 Euler's Formula

Thursday, November 11, 2010
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$$\cos A \cos B = \left(e^{jA} + e^{-jA}\right)\left(e^{jB} + e^{-jB}\right) \times \frac{1}{4}$$

$$= \left(\underbrace{e^{j(A+B)} + e^{-j(A+B)}}_{2\cos(A+B)} + \underbrace{e^{j(A-B)} + e^{-j(A-B)}}_{2\cos(A-B)}\right)\frac{1}{4}$$

$$= \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

Q3 Sinc Function and Triangular Signal

Wednesday, July 06, 2011
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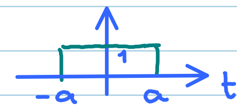
We know that



$$2a\,\mathrm{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} 1[|t| \leq a]$$ ← shown in class.

Therefore,

$$\mathrm{sinc}(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \leq a].$$

Finally,

$$\mathrm{sinc}^2(2\pi a f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \leq a] * \frac{1}{2a} 1[|t| \leq a]$$

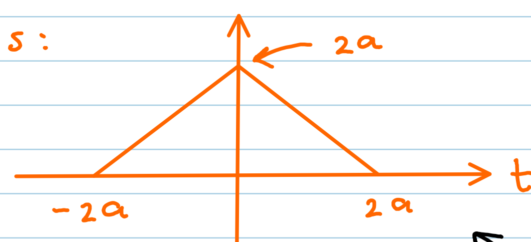
In this question, $2a = 5$.

$$= \frac{1}{4a^2}\left(1[|t| \leq a] * 1[|t| \leq a]\right)$$

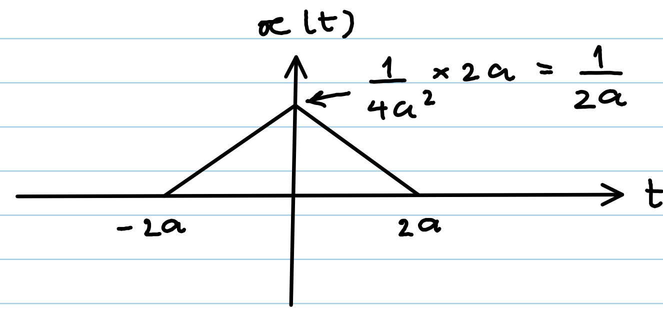
So, we can solve this question if we can find the convolution of $1[|t| \leq a]$ with itself.

This is also discussed in class:

$$1[|t| \leq a] * 1[|t| \leq a] =$$

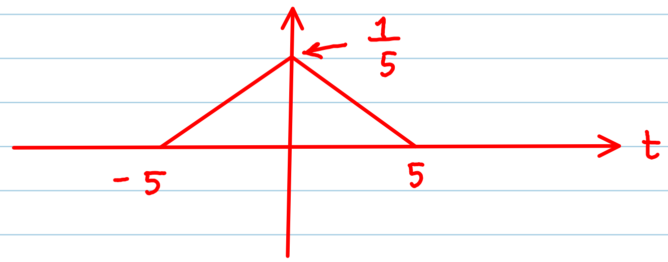


Therefore, the plot of $x(t)$ should be the same as ↑ but scaled vertically by a factor of $1/4a^2$:



For us, $a = \frac{5}{2}$. So, $2a = 5$ and the plot of $x(t)$ is

$x(t)$


1/5
-5
5
t

Q4 Manipulation of time

Wednesday, July 06, 2011
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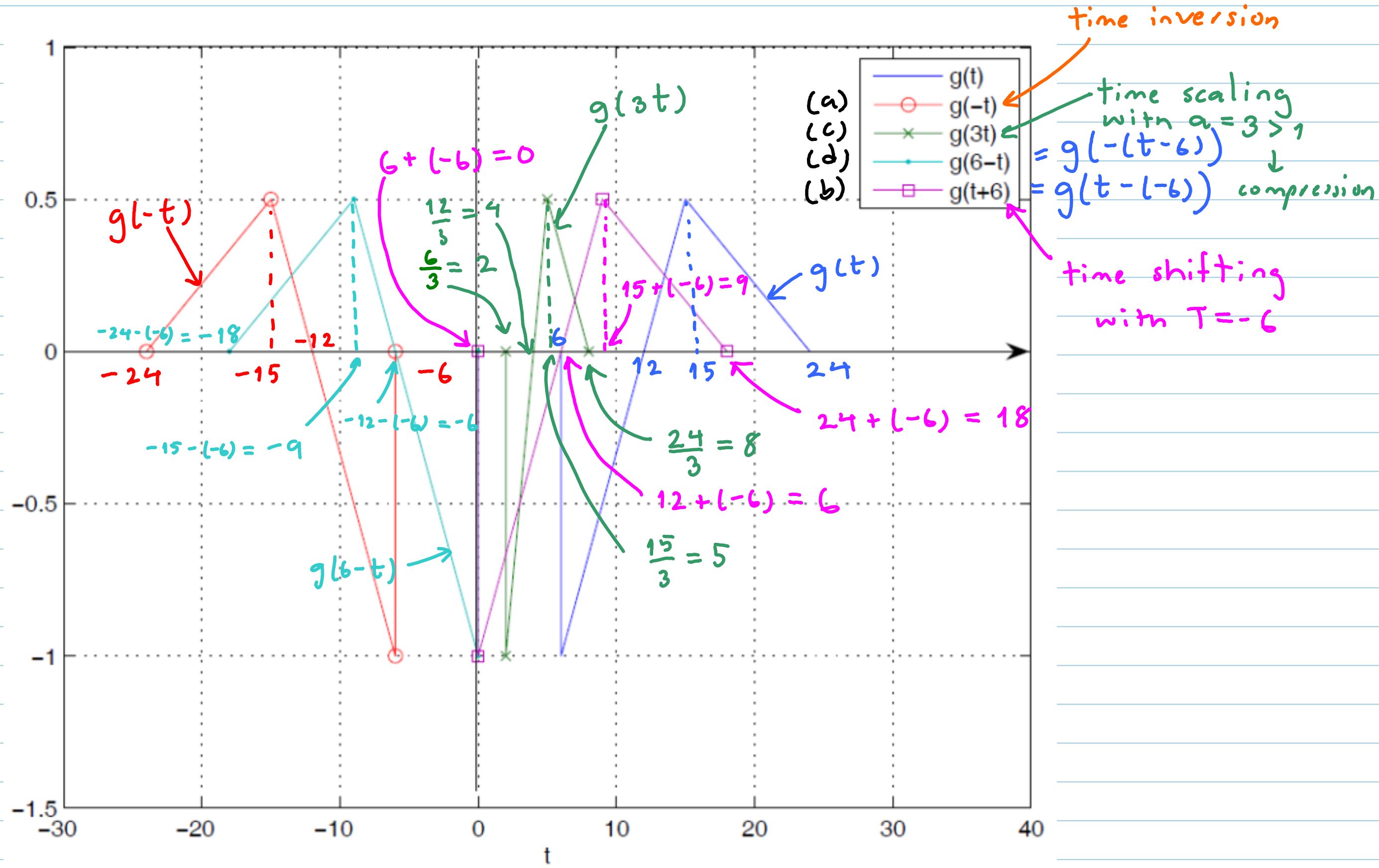
First, we review some useful signal operations

Time shifting : $g(t-T)$ represents $g(t)$ time-shifted by T.
If T is positive, the shift is to the right (delay).
If T is negative, the shift is to the left.

Time scaling : If $g(t)$ is compressed in time by a factor a $(a>1)$, the resulting signal is $g(at)$.
If $0<a<1$, the scaling is expansion.

Time inversion (Time reversal)
: $g(-t)$ is the mirror image of $g(t)$ about the vertical axis.

All the signals are plotted below



The tricky one would be $g(6-t)$.

There are two ways to think about it

time inversion   time shift, $T=6$

There are two ways to think about it

1. $g(t) \xrightarrow{\text{time inversion}} g(-t) \xrightarrow{\text{time shift, } T=6} g(-(t-6))$

mirror image about the vertical axis

shift to the right by 6

2. $g(t) \xrightarrow{\text{time shift, } T=-6} g(t+6) \xrightarrow{\text{time inversion}} g(-t+6)$

shift to the left by 6

mirror image of $g(t+6)$ about the vertical axis

Q5 Sifting Property of the Delta Function

Wednesday, July 06, 2011
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Recall the sampling property of the delta function

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(0) \qquad (\star)$$

If we let $\tau = t - T$, then $d\tau = dt$, $t = \tau + T$, and

$$(\star\star) \quad \int_{-\infty}^{\infty} \phi(t)\,\delta(t-T)\,dt = \int_{-\infty}^{\infty} \phi(\tau+T)\,\delta(\tau)\,d\tau = \phi(\tau+T)\Big|_{\tau=0} \overset{\star}{=} \phi(T)$$

Let $u = t - \tau \Rightarrow du = -d\tau$

(a) $$\int_{-\infty}^{\infty} g(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} g(t-u)\,\delta(u)\,du \overset{\star}{=} g(t-u)\Big|_{u=0} = g(t)$$

Remark: (a) and (b) simply mean $g * \delta = \delta * g = g$

(b) $g(t-\tau)\Big|_{\tau=0} = g(t)$ (use $\star$)

(c) $e^{-j2\pi f t}\Big|_{t=0} = e^0 = 1$ (use $\star$)

(d) $\sin(\pi t)\Big|_{t=2} = \sin(2\pi) = 0$ (use $\star\star$)

(e) $e^{-t}\Big|_{t=-3} = e^{-(-3)} = e^3$ (use $\star\star$)

(f) $t^3+4\Big|_{t=1} = 1^3+4 = 1+4 = 5$ (use (a))

(g) $g(2-t)\Big|_{t=3} = g(2-3) = g(-1)$ (use (a))

(h) $e^{x-1}\cos\left(\frac{\pi}{2}(x-5)\right)\Big| \quad = e^{3-1}\cos\left(\frac{\pi}{2}(3-5)\right) = e^2\cos(-\pi) = -e^2$

$t = 3$

(h) $e^{x-1}\cos\left(\frac{\pi}{2}(x-5)\right)\Big|_{x=3} = e^{3-1}\cos\left(\frac{\pi}{2}(3-5)\right) = e^{2}\cos(-\pi) = -e^{2}$

(use ★★)

(b) Note that $g_1(t) = g(-t)$.

Recall that $x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$.

Here, $a = -1$.

Therefore, $G_1(f) = \frac{1}{|-1|} G\left(\frac{f}{-1}\right) = \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right)$

(c) Note that $g_2(t) = g(t-1) + g_1(t-1)$

$$\Rightarrow \quad G_2(f) = e^{-j2\pi f} G(f) + e^{-j2\pi f} G_1(f)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1 + e^{-j\omega} + j\omega e^{-j\omega} - 1\right)$$

$$= \frac{e^{-j\omega}}{\omega^2}\left(2\cos(\omega) - j\omega(2j)\sin\omega - 2\right)$$

$$= \frac{2e^{-j2\pi f}}{(2\pi f)^2}\left(\cos(2\pi f) + 2\pi f \sin(2\pi f) - 1\right)$$

(d) Note that $g_3(t) = g(t-1) + g_1(t+1)$

$$\Rightarrow \quad G_3(f) = e^{-j2\pi f} G(f) + e^{j2\pi f} G_1(f)$$

$$= \frac{1}{\omega^2}\left(1 - \cancel{j\omega} - e^{-j\omega} + 1 + \cancel{j\omega} - e^{j\omega}\right)$$

$$= \frac{1}{\omega^2}\left(2 - 2\cos(\omega)\right) = \frac{2}{\omega^2}(1 - \cos\omega)$$

$$= \frac{2}{\omega^2} 2\sin^2\left(\frac{\omega}{2}\right) = \left(\frac{\sin\left(\frac{\omega}{2}\right)}{\omega/2}\right)^2 = \mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

Recall that

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\parallel \qquad \parallel$

$1 - \sin^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

$\Rightarrow \cos 2A = 1 - 2\sin^2 A$

$1 - \cos 2A = 2\sin^2 A$

$$= \operatorname{sinc}^2(\pi f)$$

(e) Note that $g_4(t) = g(t - \frac{1}{2}) + g_1(t + \frac{1}{2})$.

$$\Rightarrow \quad G_4(f) = e^{-j\omega/2} G(f) + e^{j\omega/2} G_1(f)$$

use part (b)

$$= e^{-j\omega/2} \frac{1}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right)$$

$$+ e^{j\omega/2} \frac{1}{\omega^2}\left(e^{-j\omega} + j\omega e^{-j\omega} - 1\right).$$

$$= \frac{1}{\omega^2}\left(\cancel{e^{j\frac{\omega}{2}}} - j\omega e^{j\frac{\omega}{2}} - \cancel{e^{-j\frac{\omega}{2}}} + \cancel{e^{-j\frac{\omega}{2}}} + j\omega e^{-j\frac{\omega}{2}} - \cancel{e^{j\frac{\omega}{2}}}\right)$$

$$= \frac{-j}{\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right) = \frac{(-j)}{\omega}(2j)\sin(\omega/2)$$

$$= \frac{\sin(\omega/2)}{\omega/2} = \operatorname{sinc}\left(\frac{\omega}{2}\right) = \operatorname{sinc}(\pi f)$$

(f) Note that $g_5(t) = 1.5\, g\left(\frac{1}{2}(t-2)\right)$

$$\Rightarrow \quad G_5(f) = 1.5 \times \frac{1}{1/2} G\left(\frac{f}{1/2}\right) e^{-j2\omega}$$

$$= 3\, G(2f)\, e^{-j}$$

$$= 3 \times \frac{1}{(2\pi 2f)^2}\left(e^{j2\omega} - j2\omega e^{j2\omega} - 1\right) e^{-j2\omega}$$

$$= \frac{3}{4\omega^2}\left(1 - 2j\omega - e^{-2j\omega}\right)$$

$$= \frac{3}{4(2\pi f)^2}\left(1 - j4\pi f - e^{-j4\pi f}\right)$$

Q7 Integrations involving sinc function(s)

Wednesday, July 27, 2011
8:23 PM

(a) We know that

$$1[|t| \le a] \xrightarrow{\mathcal{F}} 2a\,\text{sinc}(2\pi f a)$$

So,

$$1[|t| \le a] = \int_{-\infty}^{\infty} 2a\,\text{sinc}(2\pi f a)\, e^{j2\pi f t}\, df$$

Inverse transform

For $a>0$, we have

$$\int_{-\infty}^{\infty} \text{sinc}(2\pi f a)\, e^{j2\pi f t}\, df = \frac{1}{2a} 1[|t| \le a]$$

Setting $t=0$ leads to

$$\int_{-\infty}^{\infty} \text{sinc}(2\pi f a)\, df = \frac{1}{2a} = \frac{1}{\cancel{2} \times \frac{\sqrt{5}}{\cancel{2}\pi}} = \frac{\pi}{\sqrt{5}}$$

Here, $2\pi a = \sqrt{5} \Rightarrow a = \frac{\sqrt{5}}{2\pi}$

(b) Note first that $2\,\text{sinc}(2\pi f) \xrightarrow{\mathcal{F}^{-1}} 1[|t| \le 1]$ $(a=1)$

By the time-shift property,

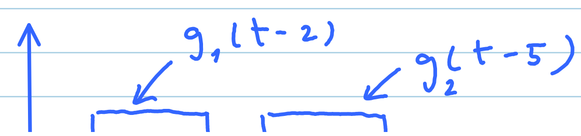
$$e^{-j2\pi f t_0}\, 2\,\text{sinc}(2\pi f) \xrightarrow{\mathcal{F}^{-1}} 1[|t-t_0| \le 1]$$

By Parseval's theorem

$$\int_{-\infty}^{\infty} \left(e^{-j2\pi f t_1} G_1(f)\right)\left(e^{-j2\pi f t_2} G_2(f)\right)^* df =$$

$$= \int_{-\infty}^{\infty} g_1(t-t_1)\, g_2^*(t-t_2)\, dt$$

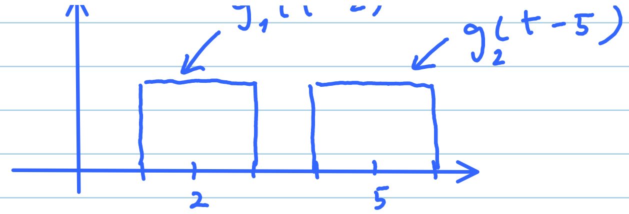
Here, $g_1(t) = g_2(t) = 1[|t| \le 1]$



$t_1 = 2$, $t_2 = 5$

Here, $g_1(t) = g_2(t) = 1[|t| \le 1]$

$t_1 = 2, \quad t_2 = 5$



No overlap, so the integral is 0.

Alternatively, we can first simplify the integral to

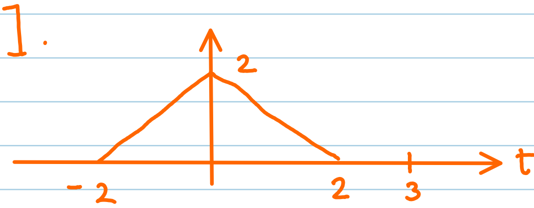
$$\int_{-\infty}^{\infty} e^{j2\pi f(t_2 - t_1)} G_1(f) G_2(f)\, df$$

This is then the inverse Fourier transform of $G_1(f) G_2(f)$ evaluated at $t = (t_2 - t_1)$.

The inverse Fourier transform is given by $g_1(t) * g_2(t)$.

Again, $g_1(t) = g_2(t) = 1[|t| \le 1]$.

So, $g_1(t) * g_2(t) =$



Here, $t_2 - t_1 = 5 - 2 = 3$. So, the integral is 0.

(c) $\mathrm{sinc}(\underbrace{2\pi a}_{=c} f) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2a} 1[|t| \le a]$.

$\Downarrow$

$a = \frac{c}{2\pi}$

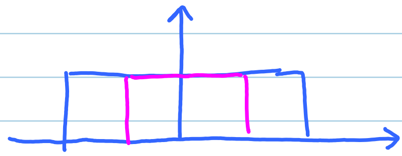
$\mathrm{sinc}(cf) \xrightarrow{\mathcal{F}^{-1}} \frac{\pi}{c} 1\left[|t| \le \frac{c}{2\pi}\right]$

Again, by Parseval's theorem,

$$\int_{-\infty}^{\infty} \mathrm{sinc}(c_1 f)\, \mathrm{sinc}^*(c_2 f)\, df = \int_{-\infty}^{\infty} \frac{\pi}{c_1} 1\left[|t| \le \frac{c_1}{2\pi}\right] \frac{\pi}{c_2} 1\left[|t| \le \frac{c_2}{2\pi}\right] dt$$

$$= \frac{\cancel{\pi^2}}{c_1 c_2} \times \frac{\min\{c_1, c_2\}}{\cancel{\pi}} = \frac{\pi}{c_1 c_2} \min\{c_1, c_2\}.$$

$$= \frac{\cancel{\pi}^2}{c_1 c_2} \times \frac{\min\{c_1, c_2\}}{\cancel{\pi}} = \frac{\pi}{c_1 c_2} \min\{c_1, c_2\}.$$



Here, $c_1 = \sqrt{5}$, $c_2 = \sqrt{7}$.

So, the integral is $\frac{\pi}{\cancel{\sqrt{5}}\sqrt{7}} \cancel{\sqrt{5}} = \frac{\pi}{\sqrt{7}}$

Alternatively, the integral is the inverse Fourier transform of $\text{sinc}(c_1 f)\,\text{sinc}(c_2 f)$ evaluated at $t = 0$.

same calculation

(d) $\text{sinc}(cf) \xrightarrow{\mathcal{F}^{-1}} \frac{\pi}{c} 1\left[|t| \le \frac{c}{2\pi}\right]$

$\Big\downarrow c = \pi$

$\text{sinc}(\pi f) \xrightarrow{\mathcal{F}^{-1}} 1\left[|t| \le \frac{1}{2}\right]$

$\text{sinc}(\pi(f - f_0)) \xrightarrow{\mathcal{F}^{-1}} e^{j2\pi f_0 t}\, 1\left[|t| \le \frac{1}{2}\right]$

By Parseval's theorem, the integral is the same as

$$\int_{-\infty}^{\infty} e^{j2\pi f_1 t}\, 1\left[|t| \le \tfrac{1}{2}\right] e^{-j2\pi f_2 t}\, 1\left[|t| \le \tfrac{1}{2}\right] dt$$

$$= \int_{-1/2}^{1/2} e^{j2\pi(f_1 - f_2)t}\, dt = \frac{1}{j2\pi(f_1 - f_2)} e^{j2\pi(f_1 - f_2)t} \Big|_{-1/2}^{1/2}$$

$$= \frac{1}{j2\pi(f_1 - f_2)} \left(e^{j2\pi(f_1 - f_2)\frac{1}{2}} - e^{-j2\pi(f_1 - f_2)\frac{1}{2}} \right)$$

$$= \frac{\sin(\pi(f_1 - f_2))}{\pi(f_1 - f_2)} = \text{sinc}(\pi(f_1 - f_2))$$

If $f_1 - f_2$ is an integer, then the integral is 0.

Here, $f_1 - f_2 = 5 - \frac{7}{2} = \frac{3}{2}$.

So, the integral is $\dfrac{\sin\left(\frac{3}{2}\pi\right)}{\frac{3}{2}\pi} = \dfrac{-1}{\frac{3}{2}\pi} = -\dfrac{2}{3\pi}$.