Name ________________________ ID3 _ _ _

ECS 332: Principles of Communications **2018/1**

HW 5 — Due: October 26, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 12 pages.

(b) (1 pt) Work and write your answers **directly on these sheets** (not on other blank sheets of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all non-optional problems.

(e) Carefully write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1. Consider the impulse train $G(f)$ shown on the right in Figure 5.1. Plot $g(t)$.

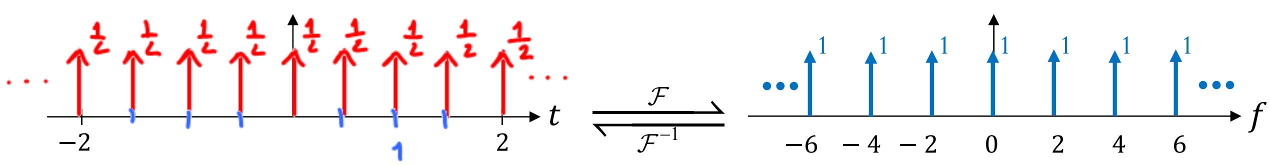


Figure 5.1: A train of impulses in the frequency domain

In class, we have seen that

$$\sum_{k=-\infty}^{\infty} \delta(t-kT_0) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} f_0\, \delta(f-kf_0) \quad \text{where } f_0 = \frac{1}{T_0}.$$

If we use only this fact, then from the size of the δ-functions in $G(f)$, we have $f_0 = 1$.

However, from the spacing btw adjacent δ-func., we have $f_0 = 2$ $\Rightarrow$ contradiction.

So, we know that the size of the δ-func. in the time domain is not "1".

By scaling the size of the δ-func. in the time domain by A, we have

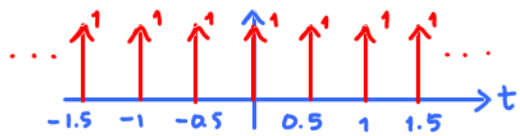
$$(*) \quad \sum_{k=-\infty}^{\infty} A\,\delta(t-kT_0) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} A f_0\, \delta(f-kf_0).$$

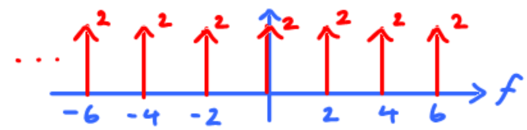
From $G(f)$,
spacing gives $f_0 = 2$. $\Rightarrow T_0 = \frac{1}{2}$
Size gives $A f_0 = 1 \Rightarrow A = \frac{1}{2}$

Problem 2. Consider a signal $r(t) = \sum_{k=-\infty}^{\infty} 2e^{j4\pi kt}$. Plot $r(t)$ and $R(f)$.
Hint: Don't try to actually plot each complex-expo. func. and add them. It is quite hopeless to determine their combination.

$r(t) = \sum_{k=-\infty}^{\infty} 2e^{j2\pi(k2)t} \xrightarrow{\mathcal{F}} R(f) = \sum_{k=-\infty}^{\infty} 2\,\delta(f - kf_0)$ where $f_0 = 2 \Rightarrow T_0 = \frac{1}{2}$

Using (*) from Problem 1, we have $Af_0 = 2 \Rightarrow A = 1$.





Problem 3. Consider a "square" wave (a train of rectangular pulses) shown in Figure 5.2. Its values periodically alternates between two values A and 0 with period T_0. At $t = 0$, its value is A.

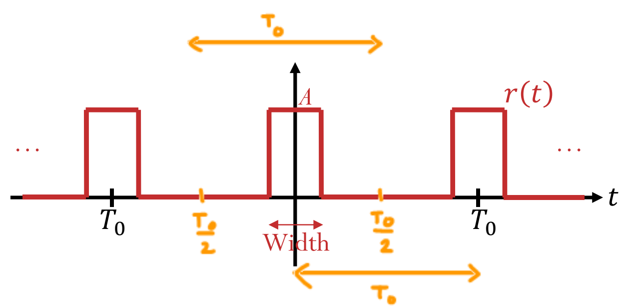


Figure 5.2: A train of rectangular pulses

First, recall that

duty cycle $= \frac{\text{width}}{T_0}$

Some values of its Fourier series coefficients are provided in the table below:

k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
c_k	$-\frac{\sqrt{2}}{7\pi}$	$-\frac{1}{3\pi}$	$-\frac{\sqrt{2}}{5\pi}$	0	$\frac{\sqrt{2}}{3\pi}$	$\frac{1}{\pi}$	$\frac{\sqrt{2}}{\pi}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{\pi}$	$\frac{1}{\pi}$	$\frac{\sqrt{2}}{3\pi}$	0	$-\frac{\sqrt{2}}{5\pi}$	$-\frac{1}{3\pi}$	$-\frac{\sqrt{2}}{7\pi}$

(a) Find its duty cycle.

In class, we've seen that when the duty cycle is $\frac{1}{n}$, the n^{th} harmonic (along with its multiples) is suppressed.

Here, $c_4 = 0$. So, we conclude that the duty cycle is $\frac{1}{4} = 25\%$.

(b) Find the value of A. (Hint: Use c_0.)

Recall that $c_0 = \frac{1}{T_0}\int_{T_0} r(t)\,dt = \underset{\text{time average.}}{\langle r(t)\rangle}$

From the picture, $\langle r(t)\rangle = \frac{\text{width} \times A}{T_0} = (\text{duty cycle}) \times A$. Therefore, $A = \frac{\langle r(t)\rangle}{\text{duty cycle}}$

We are given that $c_0 = \frac{1}{2}$ and we found, in part (a), that duty cycle $= \frac{1}{4}$.

Therefore, $A = \frac{1/2}{1/4} = 2$.

Extra Question

Here is an optional question for those who want more practice.

Problem 4 (M2011Q5)**.** In this question, you are provided with a partial proof of an important result in the study of Fourier transform. Your task is to figure out the quantities/-expressions inside the boxes labeled a,b,c, and d.

We start with a function $g(t)$. Then, we define $x(t) = \sum_{\ell=-\infty}^{\infty} g(t - \ell T)$. It is a sum that involves $g(t)$. What you will see next is our attempt to find another expression for $x(t)$ in terms of a sum that involves $G(f)$.

To do this, we first write $x(t)$ as $x(t) = g(t) * \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T)$. Then, by the convolution-in-time property, we know that $X(f)$ is given by

$$X(f) = G(f) \times \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \delta\left(f + \boxed{\qquad b \qquad}\right)$$

We can get $x(t)$ back from $X(f)$ by the inverse Fourier transform formula: $x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df$. After plugging in the expression for $X(f)$ from above, we get

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} e^{j2\pi f t} G(f) \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \delta\left(f + \boxed{\qquad b \qquad}\right) df \\ &= \boxed{\quad a \quad} \int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} e^{j2\pi f t} G(f)\, \delta\left(f + \boxed{\qquad b \qquad}\right) df. \end{aligned}$$

By interchanging the order of summation and integration, we have

$$x(t) = \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi f t} G(f)\, \delta\left(f + \boxed{\qquad b \qquad}\right) df.$$

We can now evaluate the integral via the sifting property of the delta function and get

$$x(t) = \boxed{\quad a \quad} \sum_{\ell=-\infty}^{\infty} e^{\boxed{\quad c \quad}} G\left(\boxed{\quad d \quad}\right).$$

(a) and (b) Recall that $\sum_n \delta(t - nT_0) \xrightarrow{\mathcal{F}} \frac{1}{T_0} \sum_k \delta(f - kf_0)$ where $f_0 = \frac{1}{T_0}$.

In this question, this property is applied to $\sum_\ell \delta(t - \ell T)$ to get $\sum_\ell \delta(t - \ell T) \xrightarrow{\mathcal{F}} \frac{1}{T} \sum_\ell \delta(f - \frac{\ell}{T})$

So, by the convolution-in-time rule, we have $x(t) \xrightarrow{\mathcal{F}} G(f) \times \frac{1}{T} \sum_\ell \delta(f + (-\frac{\ell}{T}))$

(c) and (d) The integral under consideration is $\int_{-\infty}^{\infty} \underbrace{e^{j2\pi f t} G(f)}_{\text{call this } b(f)} \delta(t - \frac{\ell}{T})\, df$

By the sifting property of δ-function,

$$\int_{-\infty}^{\infty} b(f)\delta(f - \tfrac{\ell}{T})\, df = b(\tfrac{\ell}{T}) = e^{j2\pi \frac{\ell}{T} t} G(\tfrac{\ell}{T})$$

Summary : $a = \frac{1}{T}$, $b = -\frac{\ell}{T}$, $c = j2\pi \frac{\ell}{T} t$, $d = \frac{\ell}{T}$