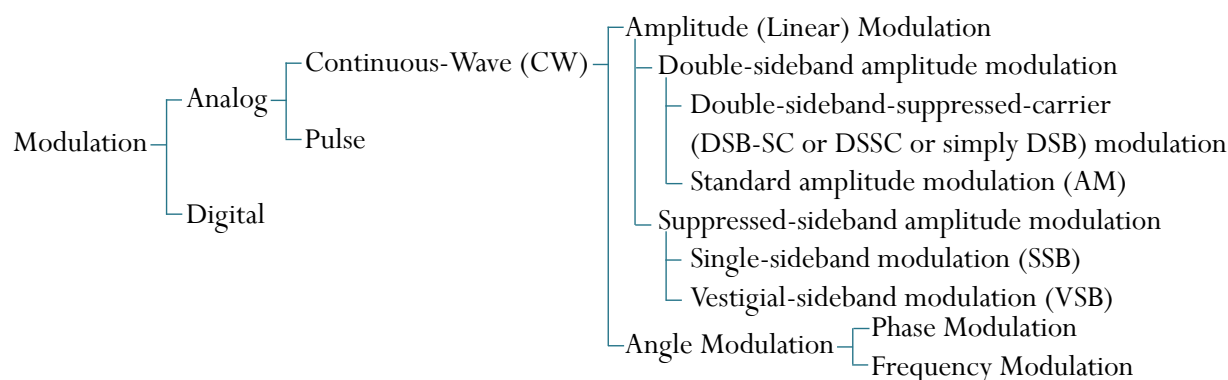


ECS332 2018/1      Part II.2      Dr.Prapun

## 4 Amplitude/Linear Modulation

4.1. The big picture:



**Definition 4.2.** A sinusoidal carrier signal  $A \cos(2\pi f_c t + \phi)$  has three basic parameters: amplitude, frequency, and phase. Varying these parameters in proportion to the baseband signal results in amplitude modulation (AM), frequency<sup>16</sup> modulation (FM), and phase modulation (PM), respectively.

Collectively, these techniques are called **continuous-wave (CW) modulation** [14, p 111][3, p 162].

<sup>16</sup>Technically, the variation of “frequency” is not as straightforward as the description here seems to suggest. For a sinusoidal carrier, a general modulated carrier can be represented mathematically as

$$x(t) = A(t) \cos(2\pi f_c t + \phi(t)).$$

Frequency modulation, as we shall see later, is resulted from letting the time derivative of  $\phi(t)$  be linearly related to the modulating signal. [15, p 112]



**Definition 4.3. Amplitude modulation** is characterized by the fact that the amplitude  $A$  of the carrier  $A \cos(2\pi f_c t + \phi)$  is varied in proportion to the baseband (message) signal  $m(t)$ .

- Because the amplitude is time-varying, we may write the modulated carrier as

$$A(t) \cos(2\pi f_c t + \phi)$$

- Because the amplitude is linearly related to the message signal, this technique is also called **linear modulation**.

#### 4.1 Double-sideband suppressed carrier (DSB-SC) modulation

**Definition 4.4.** In **double-sideband-suppressed-carrier** (DSB-SC or DSSC or simply DSB) modulation, the modulated signal is

$$x(t) = A_c \cos(2\pi f_c t) \times m(t).$$

We have seen that the multiplication by a sinusoid gives two shifted and scaled replicas of the original signal spectrum:

$$X(f) = \frac{A_c}{2} M(f - f_c) + \frac{A_c}{2} M(f + f_c).$$

- When we set  $A_c = \sqrt{2}$ , we get the “simple” modulator discussed in Example 3.13.
- As usual, we assume that the message is band-limited to  $B$ .
- We need  $f_c > B$  to avoid spectral overlapping. In practice,  $f_c \gg B$ .

**4.5. Synchronous/coherent detection by the product demodulator:** The incoming modulated signal is first multiplied with a locally generated sinusoid with the same phase and frequency (from a local oscillator (LO)) and then lowpass-filtered, the filter bandwidth being the same as the message bandwidth  $B$  or somewhat larger.

4.6. A DSB-SC modem with no channel impairment is shown in Figure 18.

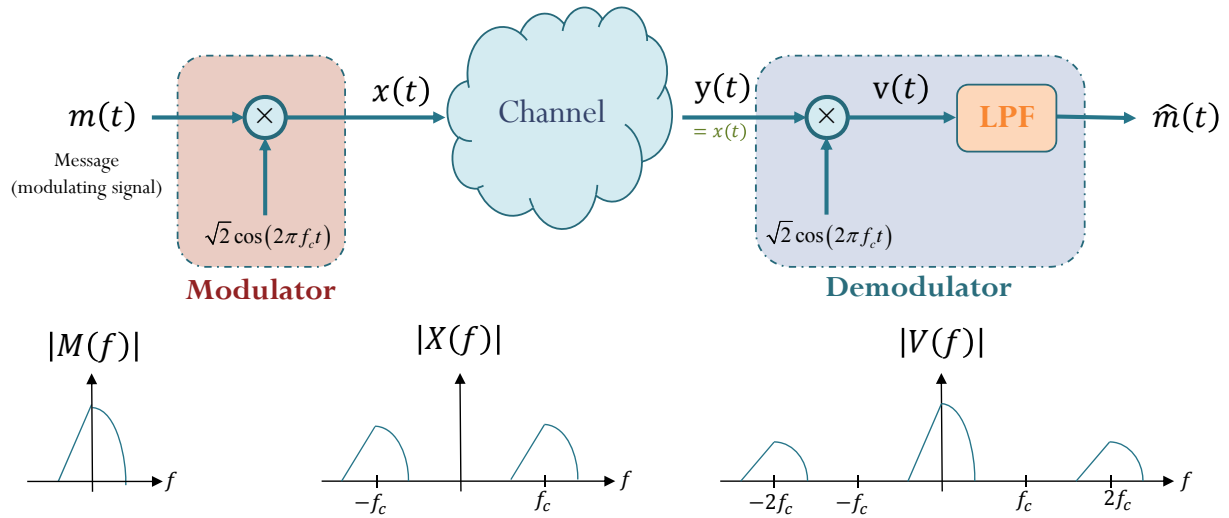


Figure 18: DSB-SC modem with no channel impairment

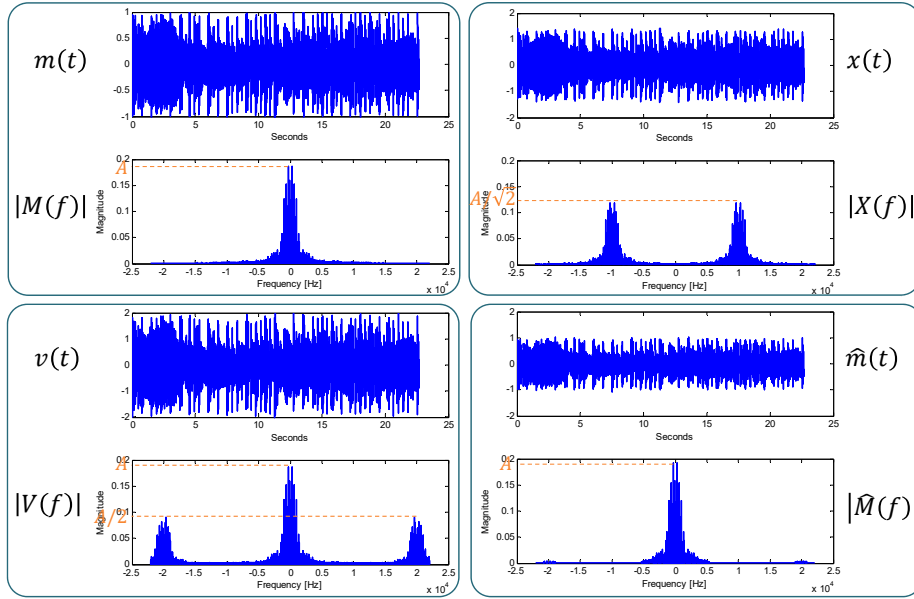


Figure 19: DSB-SC modem: signals and their spectra

Once again, recall that

$$\begin{aligned} X(f) &= \sqrt{2} \left( \frac{1}{2} (M(f - f_c) + M(f + f_c)) \right) \\ &= \frac{1}{\sqrt{2}} (M(f - f_c) + M(f + f_c)). \end{aligned}$$

Similarly,

$$\begin{aligned} v(t) &= y(t) \times \sqrt{2} \cos(2\pi f_c t) = \sqrt{2} x(t) \cos(2\pi f_c t) \\ V(f) &= \frac{1}{\sqrt{2}} (X(f - f_c) + X(f + f_c)) \end{aligned}$$

Alternatively, we can work in the time domain and utilize the trig. identity from Example 2.4:

$$\begin{aligned} v(t) &= \sqrt{2} x(t) \cos(2\pi f_c t) = \sqrt{2} \left( \sqrt{2} m(t) \cos(2\pi f_c t) \right) \cos(2\pi f_c t) \\ &= 2m(t) \cos^2(2\pi f_c t) = m(t) (\cos(2(2\pi f_c t)) + 1) \\ &= m(t) + m(t) \cos(2\pi (2f_c) t) \end{aligned}$$

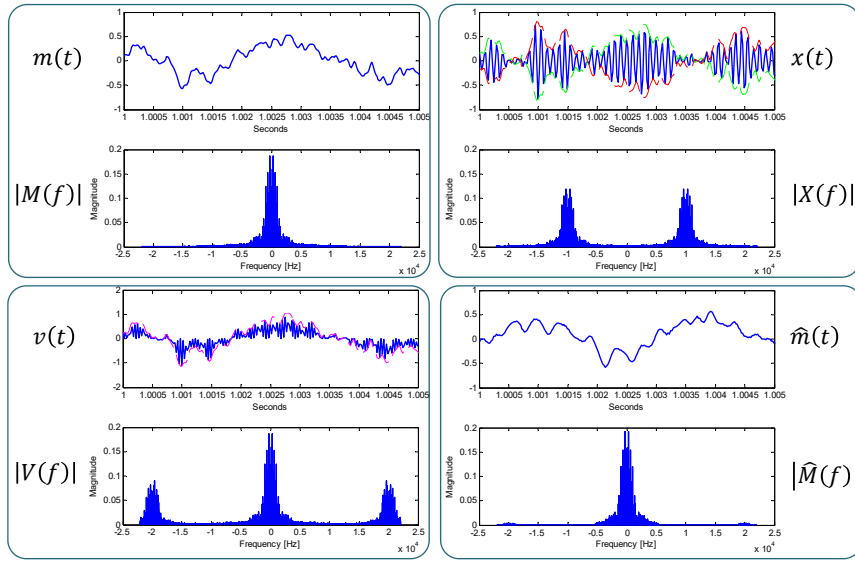


Figure 20: DSB-SC modem: signals and their spectra (zooming in)

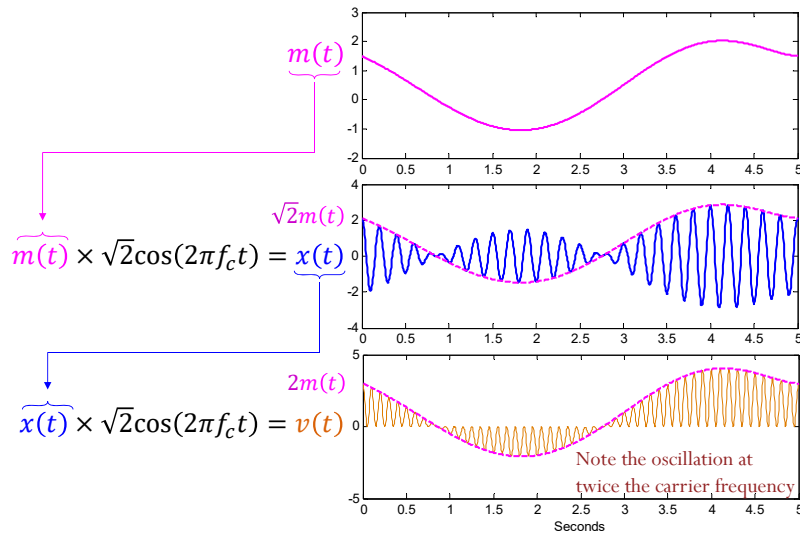


Figure 21: DSB-SC modem: signals in time domain

Key equation for DSB-SC modem:

$$\text{LPF} \left\{ \underbrace{\left( m(t) \times \sqrt{2} \cos(2\pi f_c t) \right)}_{x(t)} \times \left( \sqrt{2} \cos(2\pi f_c t) \right) \right\} = m(t), \quad (34)$$

where the frequency response of the LPF should satisfy

$$H_{\text{LP}}(f) = \begin{cases} 1, & |f| \leq B, \\ 0, & |f| \geq 2f_c - B, \\ \text{any,} & \text{otherwise.} \end{cases}$$

**4.7. Implementation issues:**

- (a) Problem 1: Modulator construction
- (b) Problem 2: Synchronization between the two (local) carriers/oscillators
- (c) Problem 3: Spectral inefficiency

**4.8. Spectral inefficiency/redundancy:** When  $m(t)$  is real-valued, its spectrum  $M(f)$  has conjugate symmetry. With such message, the corresponding modulated signal's spectrum  $X(f)$  will also inherit the symmetry but now centered at  $f_c$  (instead of at 0). The portion that lies above  $f_c$  is known as the **upper sideband** (USB) and the portion that lies below  $f_c$  is known as the **lower sideband** (LSB). Similarly, the spectrum centered at  $-f_c$  has upper and lower sidebands. Hence, this is a modulation scheme with **double sidebands**. Both sidebands contain the same (and complete) information about the message.

**4.9. Synchronization:** Observe that (34) requires that we can generate  $\cos(\omega_c t)$  both at the transmitter and receiver. This can be difficult in practice. Suppose that the frequency at the receiver is off, say by  $\Delta f$ , and that the phase is off by  $\theta$ . The effect of these frequency and phase offsets can be quantified by calculating

$$\text{LPF} \left\{ \left( m(t) \sqrt{2} \cos(2\pi f_c t) \right) \sqrt{2} \cos(2\pi (f_c + \Delta f) t + \theta) \right\},$$

which gives

$$m(t) \cos(2\pi(\Delta f)t + \theta).$$

Of course, we want  $\Delta\omega = 0$  and  $\theta = 0$ ; that is the receiver must generate a carrier in phase and frequency synchronism with the *incoming carrier*.

#### 4.10. Effect of time delay:

Suppose the propagation time is  $\tau$ , then we have

$$\begin{aligned}y(t) &= x(t - \tau) = \sqrt{2}m(t - \tau) \cos(2\pi f_c(t - \tau)) \\ &= \sqrt{2}m(t - \tau) \cos(2\pi f_c t - 2\pi f_c \tau) \\ &= \sqrt{2}m(t - \tau) \cos(2\pi f_c t - \phi_\tau).\end{aligned}$$

Consequently,

$$\begin{aligned}v(t) &= y(t) \times \sqrt{2} \cos(2\pi f_c t) \\ &= \sqrt{2}m(t - \tau) \cos(2\pi f_c t - \phi_\tau) \times \sqrt{2} \cos(2\pi f_c t) \\ &= m(t - \tau) 2 \cos(2\pi f_c t - \phi_\tau) \cos(2\pi f_c t).\end{aligned}$$

Applying the product-to-sum formula, we then have

$$v(t) = m(t - \tau) (\cos(2\pi(2f_c)t - \phi_\tau) + \cos(\phi_\tau)).$$

In conclusion, we have seen that the principle of the DSB-SC modem is based on a simple key equation (34). However, as mentioned in 4.7, there are several implementation issues that we need to address. Some solutions are provided in the subsections to follow. However, the analysis will require some knowledge of Fourier series which is reviewed in Section 4.3.



## 4.2 Energy and Power

**Definition 4.11.** For a signal  $g(t)$ , the instantaneous power  $p(t)$  dissipated in the  $1\text{-}\Omega$  resistor is  $p_g(t) = |g(t)|^2$  regardless of whether  $g(t)$  represents a voltage or a current. To emphasize the fact that this power is based upon unity resistance, it is often referred to as the **normalized (instantaneous) power**.

**Definition 4.12.** The total (normalized) **energy** of a signal  $g(t)$  is given by

$$E_g = \int_{-\infty}^{+\infty} p_g(t) dt = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \lim_{T \rightarrow \infty} \int_{-T}^T |g(t)|^2 dt.$$

**4.13.** By the **Parseval's theorem** discussed in 2.43, we have

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df.$$

**Definition 4.14.** The average (normalized) **power** of a signal  $g(t)$  is given by

$$P_g = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |g(t)|^2 dt.$$

**Definition 4.15.** To simplify the notation, there are two operators that used angle brackets to define two frequently-used integrals:

(a) The “**time-average**” operator:

$$\langle g \rangle \equiv \langle g(t) \rangle \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T g(t) dt \quad (35)$$

(b) The **inner-product** operator:

$$\langle g_1, g_2 \rangle \equiv \langle g_1(t), g_2(t) \rangle = \int_{-\infty}^{\infty} g_1(t) g_2^*(t) dt \quad (36)$$

**4.16.** Using the above definition, we may write

- $E_g = \langle g, g \rangle = \langle G, G \rangle$  where  $G = \mathcal{F}\{g\}$
- $P_g = \langle |g|^2 \rangle$

- Parseval's theorem:  $\langle g_1, g_2 \rangle = \langle G_1, G_2 \rangle$   
where  $G_1 = \mathcal{F}\{g_1\}$  and  $G_2 = \mathcal{F}\{g_2\}$

**4.17.** Time-Averaging over Periodic Signal: For *periodic* signal  $g(t)$  with period  $T_0$ , the time-average operation in (35) can be simplified to

$$\langle g \rangle = \frac{1}{T_0} \int_{T_0} g(t) dt$$

where the integration is performed over a period of  $g$ .

**Example 4.18.**  $\langle \cos(2\pi f_0 t + \theta) \rangle =$

Similarly,  $\langle \sin(2\pi f_0 t + \theta) \rangle =$

**Example 4.19.**  $\langle \cos^2(2\pi f_0 t + \theta) \rangle =$

**Example 4.20.**  $\langle e^{j(2\pi f_0 t + \theta)} \rangle = \langle \cos(2\pi f_0 t + \theta) + j \sin(2\pi f_0 t + \theta) \rangle$

**Example 4.21.** Suppose  $g(t) = ce^{j2\pi f_0 t}$  for some (possibly complex-valued) constant  $c$  and (real-valued) frequency  $f_0$ . Find  $P_g$ .

**4.22.** When the signal  $g(t)$  can be expressed in the form  $g(t) = \sum_k c_k e^{j2\pi f_k t}$  and the  $f_k$  are distinct, then its (average) power can be calculated from

$$P_g = \sum_k |c_k|^2$$

**Example 4.23.** Suppose  $g(t) = 2e^{j6\pi t} + 3e^{j8\pi t}$ . Find  $P_g$ .

**Example 4.24.** Suppose  $g(t) = 2e^{j6\pi t} + 3e^{j6\pi t}$ . Find  $P_g$ .

**Example 4.25.** Suppose  $g(t) = \cos(2\pi f_0 t + \theta)$ . Find  $P_g$ .

Here, there are several ways to calculate  $P_g$ . We can simply use Example 4.19. Alternatively, we can first decompose the cosine into complex exponential functions using the Euler's formula:

**4.26.** The (average) power of a sinusoidal signal  $g(t) = A \cos(2\pi f_0 t + \theta)$  is

$$P_g = \begin{cases} \frac{1}{2}|A|^2, & f_0 \neq 0, \\ |A|^2 \cos^2 \theta, & f_0 = 0. \end{cases}$$

This property means any sinusoid with nonzero frequency can be written in the form

$$g(t) = \sqrt{2P_g} \cos(2\pi f_0 t + \theta).$$

**4.27.** Extension of 4.26: Consider sinusoids  $A_k \cos(2\pi f_k t + \theta_k)$  whose frequencies are positive and distinct. The (average) power of their sum

$$g(t) = \sum_k A_k \cos(2\pi f_k t + \theta_k)$$

is

$$P_g = \frac{1}{2} \sum_k |A_k|^2.$$

**Example 4.28.** Suppose  $g(t) = 2 \cos(2\pi\sqrt{3}t) + 4 \cos(2\pi\sqrt{5}t)$ . Find  $P_g$ .

**Example 4.29.** Suppose  $g(t) = 3 \cos(2t) + 4 \cos(2t - 30^\circ) + 5 \sin(3t)$ . Find  $P_g$ .

**4.30.** For *periodic* signal  $g(t)$  with period  $T_0$ , there is also no need to carry out the limiting operation to find its (average) power  $P_g$ . We only need to find an average carried out over a single period:

$$P_g = \frac{1}{T_0} \int_{T_0} |g(t)|^2 dt.$$

**Example 4.31.**

**4.32.** When the Fourier series expansion (to be reviewed in Section 4.3) of the signal is available, it is easy to calculate its power:

- (a) When the corresponding Fourier series expansion  $g(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$  is known,

$$P_g = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

- (b) When the signal  $g(t)$  is real-valued and its (compact) trigonometric Fourier series expansion  $g(t) = c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(2\pi k f_0 t + \angle \phi_k)$  is known,

$$P_g = c_0^2 + 2 \sum_{k=1}^{\infty} |c_k|^2.$$

**Definition 4.33.** Based on Definitions 4.12 and 4.14, we can define three distinct classes of signals:

- (a) If  $E_g$  is finite and nonzero,  $g$  is referred to as an **energy signal**.
- (b) If  $P_g$  is finite and nonzero,  $g$  is referred to as a **power signal**.
- (c) Some signals<sup>17</sup> are neither energy nor power signals.

- Note that the power signal has infinite energy and an energy signal has zero average power; thus the two categories are disjoint.

**Example 4.34.** Rectangular pulse

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<sup>17</sup>Consider  $g(t) = t^{-1/4} 1_{[t_0, \infty)}(t)$ , with  $t_0 > 0$ .

**Example 4.35.** Sinc pulse

**Example 4.36.** For  $\alpha > 0$ ,  $g(t) = Ae^{-\alpha t}1_{[0,\infty)}(t)$  is an energy signal with  $E_g = |A|^2/2\alpha$ .

**Example 4.37.** The rotating phasor signal  $g(t) = Ae^{j(2\pi f_0 t + \theta)}$  is a power signal with  $P_g = |A|^2$ .

**Example 4.38.** The sinusoidal signal  $g(t) = A \cos(2\pi f_0 t + \theta)$  is a power signal with  $P_g = |A|^2/2$ .

**4.39.** Consider the transmitted signal

$$x(t) = m(t) \cos(2\pi f_c t + \theta)$$

in DSB-SC modulation. Suppose  $M(f - f_c)$  and  $M(f + f_c)$  do not overlap (in the frequency domain).

(a) If  $m(t)$  is a power signal with power  $P_m$ , then the average transmitted power is

$$P_x = \frac{1}{2}P_m.$$

- Q: Why is the power (or energy) reduced?

- Remark: When  $x(t) = \sqrt{2}m(t) \cos(2\pi f_c t + \theta)$  (with no overlapping between  $M(f - f_c)$  and  $M(f + f_c)$ ), we have  $P_x = P_m$ .

(b) If  $m(t)$  is an energy signal with energy  $E_m$ , then the transmitted energy is

$$E_x = \frac{1}{2}E_m.$$

**Example 4.40.** Suppose  $m(t) = \cos(2\pi f_c t)$ . Find the average power in  $x(t) = m(t) \cos(2\pi f_c t)$ .

### 4.3 Fourier Series

**Definition 4.41. Exponential Fourier series:** Let the (real or complex) signal  $r(t)$  be a *periodic* signal with period  $T_0$ . Suppose the following *Dirichlet* conditions are satisfied:

- (a)  $r(t)$  is absolutely integrable over its period; i.e.,  $\int_0^{T_0} |r(t)| dt < \infty$ .
- (b) The number of maxima and minima of  $r(t)$  in each period is finite.
- (c) The number of discontinuities of  $r(t)$  in each period is finite.

Then  $r(t)$  can be “expanded” into a linear combination of the complex exponential signals  $(e^{j2\pi(kf_0)t})_{k=-\infty}^{\infty}$  as

$$\tilde{r}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t} = c_0 + \sum_{k=1}^{\infty} \left( c_k e^{j2\pi(kf_0)t} + c_{-k} e^{-j2\pi(kf_0)t} \right) \quad (37)$$

where

$$f_0 = \frac{1}{T_0} \quad \text{and}$$

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t) e^{-j2\pi(kf_0)t} dt, \quad (38)$$

for some *arbitrary*  $\alpha$ .

We give some remarks here.

- $\tilde{r}(t) = \begin{cases} r(t), & \text{if } r(t) \text{ is continuous at } t \\ \frac{r(t^+) + r(t^-)}{2}, & \text{if } r(t) \text{ is not continuous at } t \end{cases}$

Although  $\tilde{r}(t)$  may not be exactly the same as  $r(t)$ , for the purpose of our class, it is sufficient to simply treat them as being the same (to avoid having two different notations). Of course, we need to keep in mind that unexpected results may arise at the discontinuity points.

- The parameter  $\alpha$  in the limits of the integration (38) is arbitrary. It can be chosen to simplify computation of the integral. Some references simply write  $c_k = \frac{1}{T_0} \int_{T_0} r(t) e^{-jk\omega_0 t} dt$  to emphasize that we only need to integrate over one period of the signal; the starting point is not important.



- The coefficients  $c_k$  are called the ( $k^{\text{th}}$ ) **Fourier (series) coefficients** of (the signal)  $r(t)$ . These are, in general, complex numbers.
- $c_0 = \frac{1}{T_0} \int_{T_0} r(t) dt =$  average or DC value of  $r(t)$
- The quantity  $f_0 = \frac{1}{T_0}$  is called the **fundamental frequency** of the signal  $r(t)$ . The  $k$ th multiple of the fundamental frequency (for positive  $k$ 's) is called the  $k$ th **harmonic**.
- $c_k e^{j2\pi(kf_0)t} + c_{-k} e^{-j2\pi(kf_0)t} =$  the  $k^{\text{th}}$  **harmonic component** of  $r(t)$ .  
 $k = 1 \Rightarrow$  **fundamental component** of  $r(t)$ .

**4.42.** Being able to write  $r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t}$  means we can easily find the Fourier transform of any periodic function:

$$r(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(kf_0)t} \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} R(f) =$$

The Fourier transform of any periodic function is simply a bunch of weighted delta functions occurring at multiples of the fundamental frequency  $f_0$ .

**4.43.** Formula (38) for finding the Fourier (series) coefficients

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t) e^{-j2\pi(kf_0)t} dt \quad (39)$$

is strikingly similar to formula (5) for finding the Fourier transform:

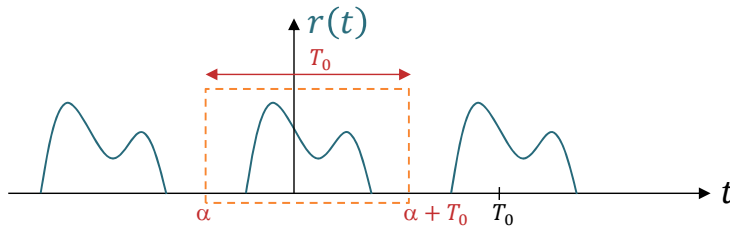
$$R(f) = \int_{-\infty}^{\infty} r(t) e^{-j2\pi ft} dt. \quad (40)$$

There are three main differences.

We have spent quite some effort learning about the Fourier transform of a signal and its properties. It would be nice to have a way to reuse those concepts with Fourier series. Identifying the three differences above lets us see their connection.

#### 4.44. Getting the Fourier coefficients from the Fourier transform:

Step 1 Consider a restricted version  $r_{T_0}(t)$  of  $r(t)$  where we only consider  $r(t)$  for one period.



Step 2 Find the Fourier transform  $R_{T_0}(f)$  of  $r_{T_0}(t)$

Step 3 *The Fourier coefficients are simply scaled samples of the Fourier transform:*

$$c_k = \frac{1}{T_0} R_{T_0}(k f_0).$$

**Example 4.45. Train of Impulses:** Find the Fourier series expansion for the train of impulses

$$\delta^{(T_0)}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

drawn in Figure 22. This infinite train of equally-spaced  $\delta$ -functions is usually denoted by the Cyrillic letter (shah).

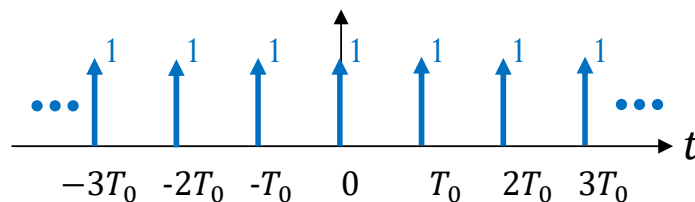
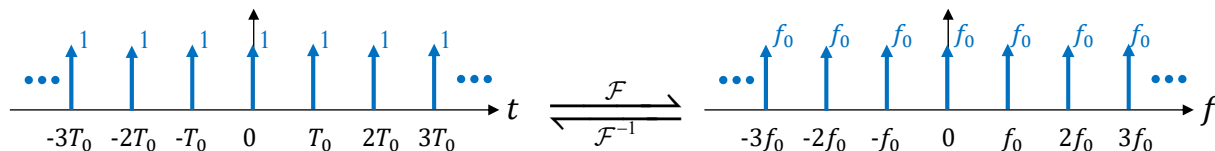


Figure 22: Train of impulses

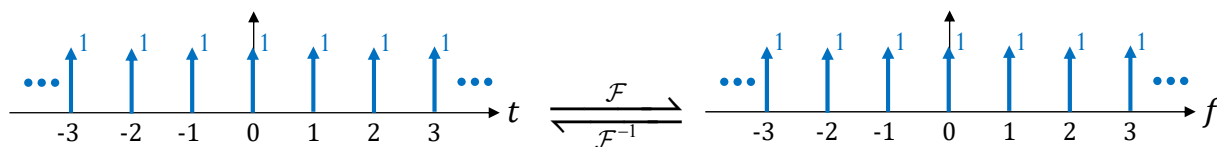
**4.46.** The Fourier series derived in Example 4.45 gives an interesting Fourier transform pair:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{j2\pi(kf_0)t} \stackrel{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftharpoons}} \quad (41)$$



A special case when  $T_0 = 1$  is quite easy to remember:

$$\sum_{n=-\infty}^{\infty} \delta(t - n) \stackrel{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftharpoons}} \sum_{k=-\infty}^{\infty} \delta(f - k) \quad (42)$$

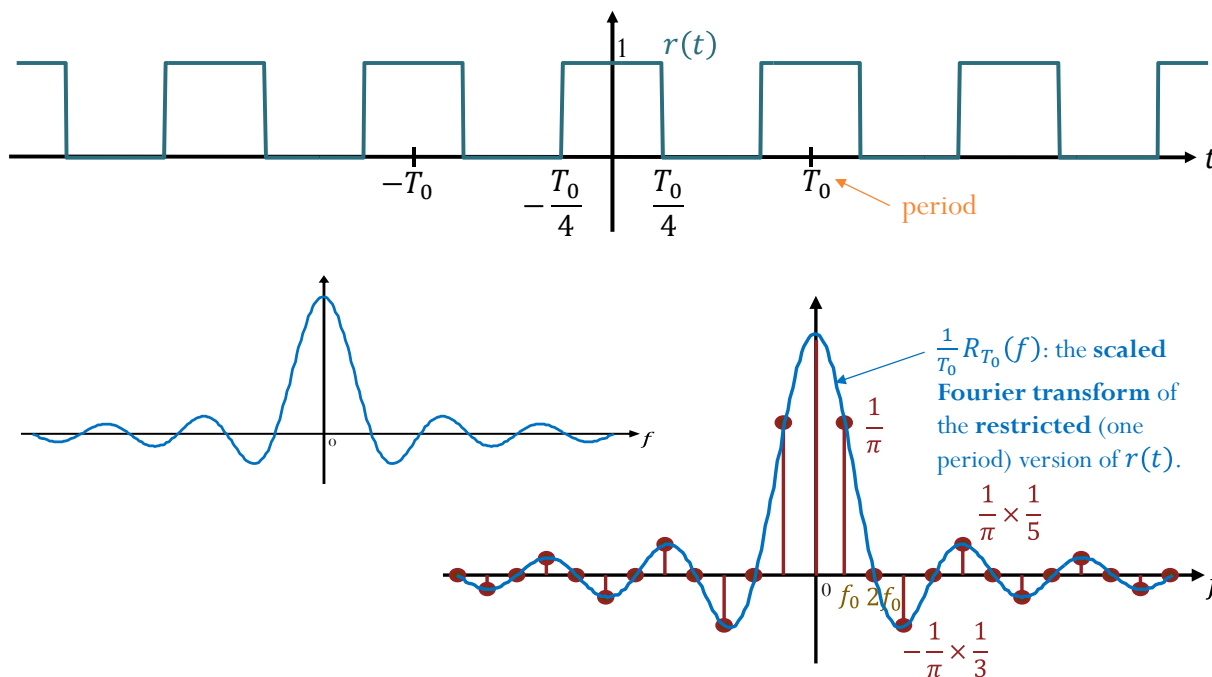


Once we remember (42), we can easily use the scaling properties of the Fourier transform (21) and the delta function (18) to generalize the special case (42) back to (41):

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \delta(at - n) &= x(at) \stackrel{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftharpoons}} \frac{1}{|a|} X\left(\frac{f}{a}\right) = \frac{1}{|a|} \sum_{k=-\infty}^{\infty} \delta\left(\frac{f}{a} - k\right) \\ &= \frac{1}{|a|} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{a}\right) \stackrel{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftharpoons}} \frac{1}{|a|} |a| \sum_{k=-\infty}^{\infty} \delta(f - ka) \\ &= \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{a}\right) \stackrel{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftharpoons}} |a| \sum_{k=-\infty}^{\infty} \delta(f - ka) \end{aligned}$$

At the end, we plug-in  $a = f_0 = 1/T_0$ .

**Example 4.47.** Find the Fourier coefficients of the **square pulse periodic signal** [6, p 57].

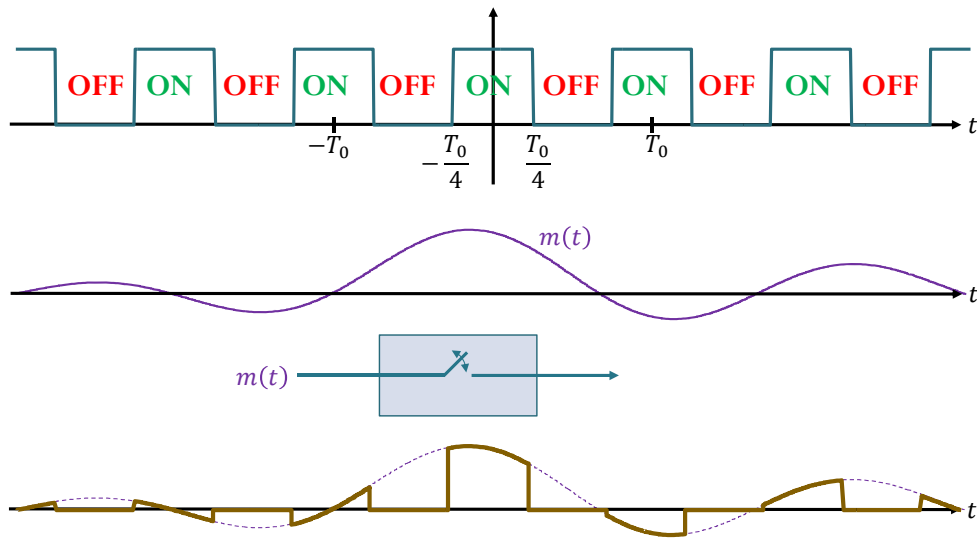


$$\begin{aligned}
 c_k &= \frac{1}{T_0} R_{T_0}(k f_0) = \frac{1}{T_0} \left( \frac{T_0}{2} \operatorname{sinc} \left( 2\pi \left( \frac{T_0}{4} \right) (f) \right) \Big|_{f=k f_0} \right) = \frac{1}{2} \operatorname{sinc} \left( k \frac{\pi}{2} \right) \\
 &= \frac{1}{2} \frac{\sin \left( k \frac{\pi}{2} \right)}{k \frac{\pi}{2}} = \frac{\sin \left( k \frac{\pi}{2} \right)}{k \pi}
 \end{aligned}$$

$k$	$k \times \frac{\pi}{2}$	$\sin \left( k \times \frac{\pi}{2} \right)$	$c_k = \frac{\sin \left( k \frac{\pi}{2} \right)}{k \pi}$	
0				
1				
2				
3				
4				
5				

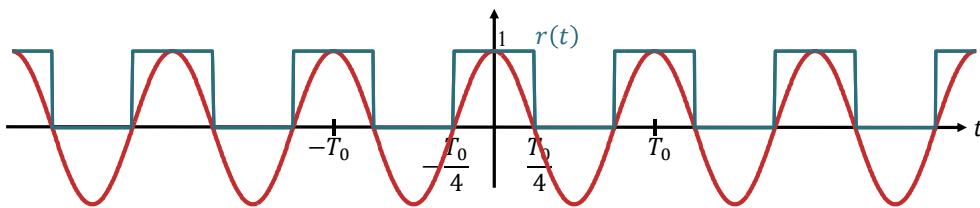
Remarks:

- (a) Multiplication by this signal is equivalent to a switching (ON-OFF) operation. (Same as periodically turning the switch on (letting another signal pass through) for half a period  $T_0$ .)



- (b) This signal can be expressed via a cosine function with the same period:

$$r(t) = 1 [\cos(2\pi f_0 t) \geq 0] = \begin{cases} 1, & \cos(2\pi f_0 t) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$



- (c) A **duty cycle** is the percentage of one period in which a signal is “active”. Here,

$$\text{duty cycle} = \text{proportion of the “ON” time} = \frac{\text{width}}{\text{period}}.$$

In this example, the duty cycle is  $\frac{T_0/2}{T_0} = 50\%$ . When the duty cycle is  $\frac{1}{n}$ , the  $n$ th harmonic ( $c_n$ ) along with its nonzero multiples are suppressed.

**4.48.** Parseval's Identity:  $P_r = \langle |r(t)|^2 \rangle = \frac{1}{T_0} \int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$ .

**4.49. Fourier series expansion for real valued function:** Suppose  $r(t)$  in the previous section is real-valued; that is  $r^* = r$ . Then, we have  $c_{-k} = c_k^*$  and we provide here three alternative ways to represent the Fourier series expansion:

$$\tilde{r}(t) = \sum_{k=-\infty}^{\infty} c_n e^{j2\pi k f_0 t} = c_0 + \sum_{k=1}^{\infty} (c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t}) \quad (43a)$$

$$= c_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k f_0 t)) + \sum_{k=1}^{\infty} (b_k \sin(2\pi k f_0 t)) \quad (43b)$$

$$= c_0 + 2 \sum_{k=1}^{\infty} |c_k| \cos(2\pi k f_0 t + \angle c_k) \quad (43c)$$

where the corresponding coefficients are obtained from

$$c_k = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} r(t) e^{-j2\pi k f_0 t} dt = \frac{1}{2} (a_k - j b_k) \quad (44)$$

$$a_k = 2\text{Re}\{c_k\} = \frac{2}{T_0} \int_{T_0} r(t) \cos(2\pi k f_0 t) dt \quad (45)$$

$$b_k = -2\text{Im}\{c_k\} = \frac{2}{T_0} \int_{T_0} r(t) \sin(2\pi k f_0 t) dt \quad (46)$$

$$2|c_k| = \sqrt{a_k^2 + b_k^2} \quad (47)$$

$$\angle c_k = -\arctan\left(\frac{b_k}{a_k}\right) \quad (48)$$

$$c_0 = \frac{a_0}{2} \quad (49)$$

The Parseval's identity can then be expressed as

$$P_r = \langle |r(t)|^2 \rangle = \frac{1}{T_0} \int_{T_0} |r(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 = c_0^2 + 2 \sum_{k=1}^{\infty} |c_k|^2$$

**4.50.** To go from (43a) to (43b) and (43c), note that when we replace  $c_{-k}$  by  $c_k^*$ , we have

$$\begin{aligned} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t} &= c_k e^{j2\pi k f_0 t} + c_k^* e^{-j2\pi k f_0 t} \\ &= c_k e^{j2\pi k f_0 t} + (c_k e^{j2\pi k f_0 t})^* \\ &= 2 \operatorname{Re} \{ c_k e^{j2\pi k f_0 t} \}. \end{aligned}$$

- Expression (43c) then follows directly from the phasor concept:

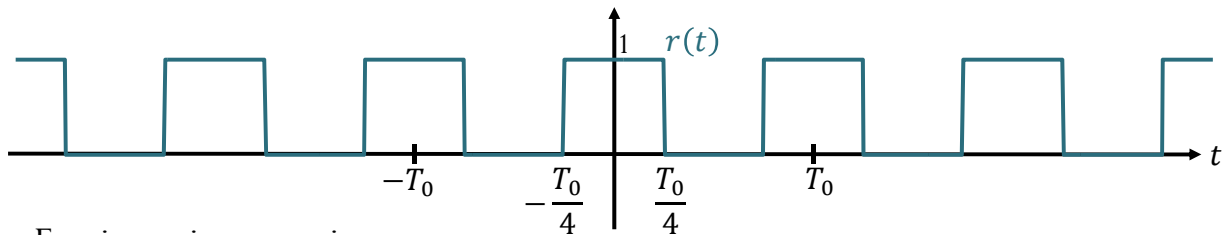
$$\operatorname{Re} \{ c_k e^{j2\pi k f_0 t} \} = |c_k| \cos(2\pi k f_0 t + \angle c_k).$$

- To get (43b), substitute  $c_k$  by  $\operatorname{Re} \{ c_k \} + j \operatorname{Im} \{ c_k \}$  and  $e^{j2\pi k f_0 t}$  by  $\cos(2\pi k f_0 t) + j \sin(2\pi k f_0 t)$ .

**Example 4.51.** For the train of impulses in Example 4.45,

$$\sum_{n=-\infty}^{\infty} \delta(t - n) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} e^{j2\pi(kf_0)t} = \frac{1}{T_0} + \frac{2}{T_0} \sum_{k=1}^{\infty} \cos k\omega_0 t \quad (50)$$

**Example 4.52.** For the rectangular pulse train in Example 4.47,



Fourier series expansion:

$$\begin{aligned} r(t) &= \frac{1}{2} + \frac{1}{\pi} e^{j(2\pi f_0 t)} - \frac{1}{3\pi} e^{j(2\pi(3f_0)t)} + \frac{1}{5\pi} e^{j(2\pi(5f_0)t)} + \dots \\ &\quad + \frac{1}{\pi} e^{j(2\pi(-f_0)t)} - \frac{1}{3\pi} e^{j(2\pi(-3f_0)t)} + \frac{1}{5\pi} e^{j(2\pi(-5f_0)t)} + \dots \end{aligned}$$

Trigonometric Fourier series expansion:

$$r(t) = \frac{1}{2} + \frac{2}{\pi} \cos(2\pi f_0 t) - \frac{2}{3\pi} \cos(2\pi(3f_0)t) + \frac{2}{5\pi} \cos(2\pi(5f_0)t) + \dots$$

$$e^{jx} + e^{-jx} = 2\cos(x)$$

$$1 [\cos \omega_0 t \geq 0] = \frac{1}{2} + \frac{2}{\pi} \left( \cos \omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t - \frac{1}{7} \cos 7\omega_0 t + \dots \right) \quad (51)$$

**Example 4.53.** Bipolar square pulse periodic signal [6, p 59]:

$$\text{sgn}(\cos \omega_0 t) = \frac{4}{\pi} \left( \cos \omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t - \frac{1}{7} \cos 7\omega_0 t + \dots \right)$$

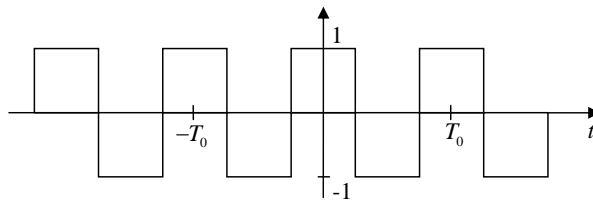


Figure 23: Bipolar square pulse periodic signal