ECS 315: Probability and Random Processes **2019/1**

HW Solution 8 — Due: October 31, 4 PM

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Problem 1. [F2013/1] For each of the following random variables, find $P[1 < X \le 2]$.

(a) $X \sim \text{Binomial}(3, 1/3)$

(b) $X \sim \text{Poisson}(3)$

Solution:

(a) Because $X \sim \text{Binomial}(3, 1/3)$, we know that X can only take the values 0, 1, 2, 3. Only the value 2 satisfies the condition given. Therefore, $P[1 < X \le 2] = P[X = 2] = p_X(2)$. Recall that the pmf for the binomial random variable is

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for $x = 0, 1, 2, 3, \ldots, n$. Here, it is given that $n = 3$ and $p = 1/3$. Therefore,

$$p_X(2) = \binom{3}{2} \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right)^{3-2} = 3 \times \frac{1}{9} \times \frac{2}{3} = \boxed{\frac{2}{9}}.$$

(b) Because $X \sim \text{Poisson}(3)$, we know that X can take the values 0, 1, 2, 3, ... As in the previous part, only the value 2 satisfies the condition given. Therefore, $P[1 < X \le 2] = P[X = 2] = p_X(2)$. Recall that the pmf for the Poisson random variable is

$$p_X(x) = e^{-\alpha} \frac{\alpha^x}{x!}$$

for $x = 0, 1, 2, 3, \ldots$. Here, it is given that $\alpha = 3$. Therefore,

$$p_X(2) = e^{-3} \frac{3^2}{2!} = \boxed{\frac{9}{2} e^{-3} \approx 0.2240}.$$

Problem 2. Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 2$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:05. What is the probability that $M < 2$?

Solution: Here, we are given that $M \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 2 \times 5 = 10$. Recall that, for $M \sim \mathcal{P}(\alpha)$, we have

$$P[M = m] = \begin{cases} e^{-\alpha} \frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$P[M < 2] = P[M = 0] + P[M = 1] = e^{-\alpha}\frac{\alpha^0}{0!} + e^{-\alpha}\frac{\alpha^1}{1!}$$
$$= e^{-\alpha}(1+\alpha) = e^{-10}(1+10) = 11e^{-10} \approx 5 \times 10^{-4}.$$

Problem 3. [M2011/1] The cdf of a random variable X is plotted in Figure 8.1.

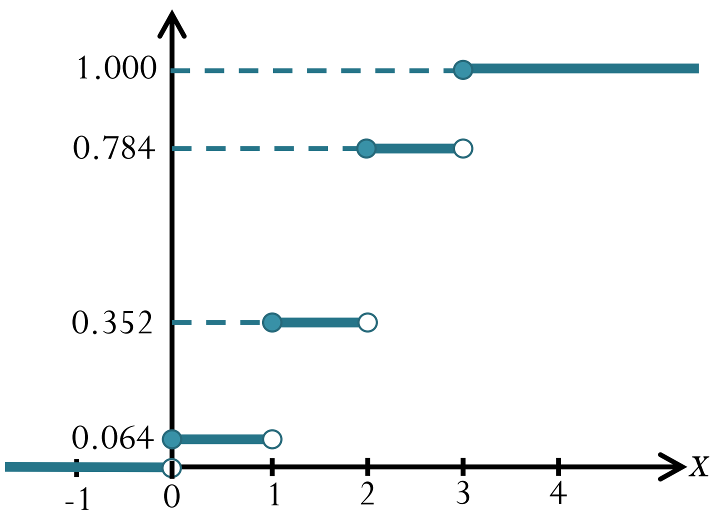

Figure 8.1: CDF of X for Problem 3

(a) Find the pmf $p_X(x)$.

(b) Find the family to which X belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

Solution:

(a) For discrete random variable, $P[X = x]$ is the jump size at x on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.

- $P[X = 0]$ = the jump size at $0 = 0.064 = \frac{64}{1000} = (4/10)^3 = (2/5)^3$.
- $P[X = 1]$ = the jump size at $1 = 0.352 - 0.064 = 0.288$.
- $P[X = 2]$ = the jump size at $2 = 0.784 - 0.352 = 0.432$.
- $P[X = 3]$ = the jump size at $3 = 1 - 0.784 = 0.216 = (6/10)^3$.

In conclusion,

$$\boxed{p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) Among all the pmf that we discussed in class, only binomial pmf can have support $= \{0, 1, 2, 3\}$ with unequal probabilities. To check that the RV really is binomial, recall that the pmf for binomial X is given by $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ for $x = 0, 1, 2, \ldots, n$. Here, $n = 3$. Furthermore, observe that $p_X(0) = (1-p)^n$. By comparing $p_X(0)$ with what we had in part (a), we have $1 - p = 2/5$ or $p = 3/5$. For $x = 1, 2, 3$, plugging in $p = 3/5$ and $n = 3$ in to $p_X(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$ gives the same values as what we had in part (a). So, $\boxed{X \sim \mathcal{B}\left(3, \frac{3}{5}\right)}$.

Problem 4. When n is large, binomial distribution Binomial(n, p) becomes difficult to compute directly . In this question, we will consider an approximation when the value of p is close to 0. In such case, the binomial can be approximated[1] by the Poisson distribution with parameter $\alpha = np$. For this approximation to work, we will see in this exercise that n does not have to be very large and p does not need to be very small.

(a) Let $X \sim$ Binomial$(12, 1/36)$. (For example, roll two dice 12 times and let X be the number of times a double 6 appears.) Evaluate $p_X(x)$ for $x = 0, 1, 2$.

(b) Compare your answers part (a) with its Poisson approximation.

Solution:

(a) For Binomial(n, p) random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that $n = 12$ and $p = \frac{1}{36}$. Plugging in $x = 0, 1, 2$, we get $\boxed{0.7132, 0.2445, 0.0384}$, respectively

[1] More specifically, suppose X_n has a binomial distribution with parameters n and p_n. If $p_n \to 0$ and $np_n \to \alpha$ as $n \to \infty$, then

$$P[X_n = k] \to e^{-\alpha} \frac{\alpha^k}{k!}.$$

(b) A Poisson random variable with parameter $\alpha = np$ can approximate a Binomial(n, p) random variable when n is large and p is small. Here, with $n = 12$ and $p = \frac{1}{36}$, we have $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$. The Poisson pmf at $x = 0, 1, 2$ is given by $e^{-\alpha}\frac{\alpha^x}{x!} = e^{-1/3}\frac{(1/3)^x}{x!}$. Plugging in $x = 0, 1, 2$ gives $\boxed{0.7165, 0.2388, 0.0398}$, respectively.

Figure **??** compares the two pmfs. Note how close they are!

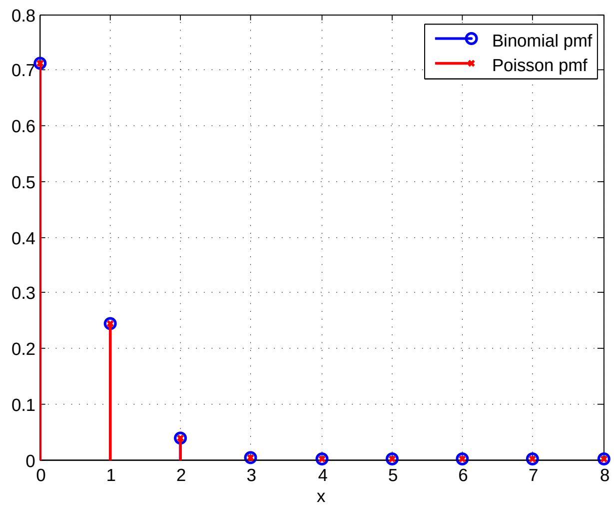

Figure 8.2: Poisson Approximation

Problem 5. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

Solution: Let N be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering $n = 500$ trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is $p = \frac{1}{365}$. Then, this $N \sim$ Binomial(n, p).

(a) Binomial: $P[N = 1] = np^1(1 - p)^{n-1} \approx \boxed{0.348}$.

(b) Poisson: $P[N = 1] = e^{-np}\frac{(np)^1}{1!} \approx \boxed{0.348}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 6. A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

(a) exactly one is emitted,

(b) more than three are emitted,

(c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

Solution: Let X be the number or particles emitted during the one second under consideration. Then $X \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T = 0.7 \times 1 = 0.7$.

(a) $P[X = 1] = e^{-\alpha}\frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx \boxed{0.3477}$.

(b) $P[X > 3] = 1 - P[X \leq 3] = 1 - \sum_{k=0}^{3} e^{-0.7}\frac{0.7^k}{k!} \approx \boxed{0.0058}$.

(c) $P[1 \leq X \leq 4] = \sum_{k=1}^{4} e^{-0.7}\frac{0.7^k}{k!} \approx \boxed{0.5026}$.

Problem 7 (M2011/1)**.** You are given an unfair coin with probability of obtaining a heads equal to $1/3,000,000,000$. You toss this coin 6,000,000,000 times. Let A be the event that you get "tails for all the tosses". Let B be the event that you get "heads for all the tosses".

(a) Approximate $P(A)$.

(b) Approximate $P(A \cup B)$.

Solution: Let N be the number of heads among the n tosses. Then, $N \sim \mathcal{B}(n, p)$. Here, we have small $p = 1/3 \times 10^9$ and large $n = 6 \times 10^9$. So, we can apply Poisson approximation. In other words, $\mathcal{B}(n, p)$ is well-approximated by $\mathcal{P}(\alpha)$ where $\alpha = np = 2$.

(a) $P(A) = P[N = 0] = e^{-2}\frac{2^0}{0!} = \frac{1}{e^2} \approx \boxed{0.1353}$.

(b) Note that events A and B are disjoint. Therefore, $P(A \cup B) = P(A) + P(B)$. We have already calculated $P(A)$ in the previous part. For $P(B)$, from $N \sim \mathcal{B}(n, p)$, we have $P(B) = P[N = n] = p^n = \left(\frac{1}{3\times 10^9}\right)^{6\times 10^9}$. Observe that $P(B)$ is extremely small compared to $P(A)$. Therefore, $P(A \cup B)$ is approximately the same as $P(A) \approx \boxed{0.1353}$.