ECS 315: Probability and Random Processes **2019/1**

HW Solution 2 — Due: September 5, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. [Montgomery and Runger, 2010, Q2-54] Each of the possible five outcomes of a random experiment is equally likely. The sample space is $\{a, b, c, d, e\}$. Let A denote the event $\{a, b\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

Solution: Because the outcomes are equally likely, we can simply use classical probability.

(a) $P(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{2}{5}}$

(b) $P(B) = \frac{|B|}{|\Omega|} = \boxed{\frac{3}{5}}$

(c) $P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{5-2}{5} = \boxed{\frac{3}{5}}$

(d) $P(A \cup B) = \frac{|\{a,b,c,d,e\}|}{|\Omega|} = \frac{5}{5} = \boxed{1}$

(e) $P(A \cap B) = \frac{|\emptyset|}{|\Omega|} = \boxed{0}$

Problem 2. (**Classical Probability and Combinatorics**) Shuffle a deck of cards and cut it into three piles. What is the probability that (at least) a court card will turn up on top of one of the piles.

Hint: There are 12 court cards (four jacks, four queens and four kings) in the deck.

Solution: In [Lovell, 2006, p. 17–19], this problem is named "Three Lucky Piles".

Method 1: When somebody cuts three piles, they are, in effect, randomly picking three cards from the deck. There are $52 \times 51 \times 50$ possible outcomes. The number of outcomes that do not contain any court card is $40 \times 39 \times 38$. So, the probability of having at least one court card is

$$\boxed{\frac{52 \times 51 \times 50 - 40 \times 39 \times 38}{52 \times 51 \times 50} \approx 0.553.}$$

Method 2: Note that our solution above, especially the part where we use the words "in effect", may not be so evident to some of you. If you want to solve this question directly, you can approach it using the total probability theorem which is studied in Chapter 6. In the beginning, we shuffle the cards. So, after the shuffling, we will have a deck of 52 cards with all the possible 52! permutations being equally likely. (In our mind,) we label the cards with #1 to #52 from the top to bottom. Now, the next step is to cut it into three piles. Note that this is the same as choosing two cards (from #2 (top) to #52 (bottom)) to indicate where the two boundaries (which are the same as the two cards at the top of second and third piles) are. Note also that this process is usually biased. Most will try to divide the deck into three piles of approximately equal size. So, it is *unlikely* that you will have the first piles with 50 cards, the second with only one card, and the third with only one card. So, classical probability can not be used here. We only know that there are $\binom{51}{2} = 1,275$ ways to perform the cutting for a particular deck and they are not equally likely. Let event $B_1, \dots, B_{1275}$ denote each of these cases. For example, B_{134} may be the case in which the cutting positions are at cards #32 and #45. So, the top cards on the three piles are cards #1, #32, and #45. Let A be the event that at least one of these cards is a court card. Of course, the "at least one" counting problem can be simplified by considering the opposite case. A^c is the event that none of the three top cards is a court card. So, there are $52 - 12 = 40$ choices for card #1. There are $40 - 1 = 39$ choices for card #32. There are $39 - 1 = 38$ choices for card #45. For the remaining $52 - 3 = 49$ cards, there is no restriction. So, there are 49! choices. In total, we have $40 \times 39 \times 38 \times (49!)$ shuffled patterns among the 52! equally likely possibilities that satisfy A^c. Therefore,

$$P(A|B_{134}) = \frac{52! - 40 \times 39 \times 38 \times (49!)}{52!} = 1 - \frac{40 \times 39 \times 38}{52 \times 51 \times 50} \approx 0.553.$$

The same reasoning applies to any cutting positions. So, $P(A|B_k) \approx 0.553$ for any k. By the total probability theorem,

$$P(A) = \sum_{k=1}^{1275} P(A|B_k)\,P(B_k) \approx \sum_{k=1}^{1275} 0.553 P(B_k) = 0.553 \sum_{k=1}^{1275} P(B_k) = 0.553 \times 1 = 0.553.$$

Observe that we still don't know the value of each $P(B_k)$ but we know that the sum of them is 1.

Problem 3. (**Classical Probability**) There are three buttons which are painted red on one side and white on the other. If we tosses the buttons into the air, calculate the probability that all three come up the same color.

Remarks: A *wrong* way of thinking about this problem is to say that there are four ways they can fall. All red showing, all white showing, two reds and a white or two whites and a red. Hence, it seems that out of four possibilities, there are two favorable cases and hence the probability is 1/2.

Solution: There are 8 possible outcomes. (The same number of outcomes as tossing three coins.) Among these, only two outcomes will have all three buttons come up the same color. So, the probability is $2/8 = \boxed{1/4}$.

Problem 4. (**Classical Probability and Combinatorics**) A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases.

(a) How many different designs are possible? [Montgomery and Runger, 2010, Q2-51]

(b) A specific design is randomly generated by the Web server when you visit the site. If you visit the site five times, what is the probability that you will not see the same design? [Montgomery and Runger, 2010, Q2-71]

Solution:

(a) By the multiplication rule, total number of possible designs

$$= 4 \times 3 \times 5 \times 3 \times 5 = \boxed{900}.$$

(b) From part (a), total number of possible designs is 900. The sample space is now the set of all possible designs that may be seen on five visits. It contains $(900)^5$ outcomes. (This is ordered sampling with replacement.)

The number of outcomes in which all five visits are different can be obtained by realizing that this is ordered sampling without replacement and hence there are $(900)_5$ outcomes. (Alternatively, On the first visit any one of 900 designs may be seen. On the second visit there are 899 remaining designs. On the third visit there are 898 remaining designs. On the fourth and fifth visits there are 897 and 896 remaining designs, respectively. From the multiplication rule, the number of outcomes where all designs are different is $900 \times 899 \times 898 \times 897 \times 896$.)

Therefore, the probability that a design is not seen again is

$$\frac{(900)_5}{900^5} \approx \boxed{0.9889.}$$

Problem 5. (**Classical Probability and Combinatorics**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

Solution: The number of ways to select two parts from 50 is $\binom{50}{2}$ and the number of ways to select two defective parts from the 5 defective ones is $\binom{5}{2}$ Therefore the probability is

$$\frac{\binom{5}{2}}{\binom{50}{2}} = \frac{2}{245} = \boxed{0.0082}.$$

Alternatively, if the two parts in the sample are selected one by one, then we may also consider their ordering as well. In such case, we use the formula for "ordered sampling without replacement" instead of "unordered sampling without replacement":

$$\frac{(5)_2}{(50)_2} = \frac{5 \times 4}{50 \times 49} = \frac{2}{245} = \boxed{0.0082}.$$

Problem 6. (**Combinatorics**) Consider the design of a communication system in the United States.

(a) How many three-digit phone prefixes that are used to represent a particular geographic area (such as an area code) can be created from the digits 0 through 9?

(b) How many three-digit phone prefixes are possible in which no digit appears more than once in each prefix?

(c) As in part (a), how many three-digit phone prefixes are possible that do not start with 0 or 1, but contain 0 or 1 as the middle digit?

[Montgomery and Runger, 2010, Q2-45]

Solution:

(a) From the multiplication rule (or by realizing that this is ordered sampling with replacement), $10^3 = \boxed{1,000}$ prefixes are possible

(b) This is ordered sampling without replacement. Therefore $(10)_3 = 10 \times 9 \times 8 = \boxed{720}$ prefixes are possible

(c) From the multiplication rule, $8 \times 2 \times 10 = \boxed{160}$ prefixes are possible.

Problem 7. (**Classical Probability and Combinatorics**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

Solution: There are 2^{10} possible outcomes for ten coin tosses. (For each toss, there is two possibilities, H or T). Only $\binom{10}{5}$ among these outcomes have exactly heads and five tails. (Choose 5 positions from 10 position for H. Then, the rest of the positions are automatically T.) The probability of have exactly 5 H and 5 T is

$$\binom{5}{5}\ \boxed{\frac{\binom{10}{5}}{2^{10}} \approx 0.246.}$$

Note that five heads and five tails will turn up more frequently than any other single combination (one head, nine tails for example) but the sum of all the other possibilities is much greater than the single 5 H, 5 T combination.

Extra Question

Here is an optional question for those who want more practice.

Problem 8. *An Even Split at Coin Tossing*: Let p_n be the probability of getting exactly n heads (and hence exactly n tails) when a fair coin is tossed $2n$ times.

(a) Find p_n.

(b) Sometimes, to work theoretically with large factorials, we use Stirling's Formula:

$$n! \approx \sqrt{2\pi n}n^n e^{-n} = \left(\sqrt{2\pi e}\right) e^{\left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)}. \tag{2.1}$$

Approximate p_n using Stirling's Formula.

(c) Find $\lim_{n\to\infty} p_n$.

Solution: Note that we have worked on a particular case ($n = 5$) of this problem earlier.

(a) Use the same solution as Problem 7; change 5 to n and 10 to $2n$, we have

$$p_n = \boxed{\frac{\binom{2n}{n}}{2^{2n}}}. = \frac{\frac{(2n)!}{n!\,n!}}{2^{2n}}$$

(b) By Stirling's Formula, we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \approx \frac{\sqrt{2\pi 2n}(2n)^{2n}e^{-2n}}{\left(\sqrt{2\pi n}n^n e^{-n}\right)^2} = \frac{4^n}{\sqrt{\pi n}}.$$

Hence,

$$p_n \approx \boxed{\frac{1}{\sqrt{\pi n}}}. \tag{2.2}$$

[Mosteller, *Fifty Challenging Problems in Probability with Solutions*, 1987, Problem 18]

See Figure **??** for comparison of p_n and its approximation via Stirling's formula.

(c) From (**??**), $\lim_{n\to\infty} p_n = \boxed{0}$. A more rigorous proof of this limit would use the bounds

$$\frac{4^n}{\sqrt{4n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n+1}}.$$

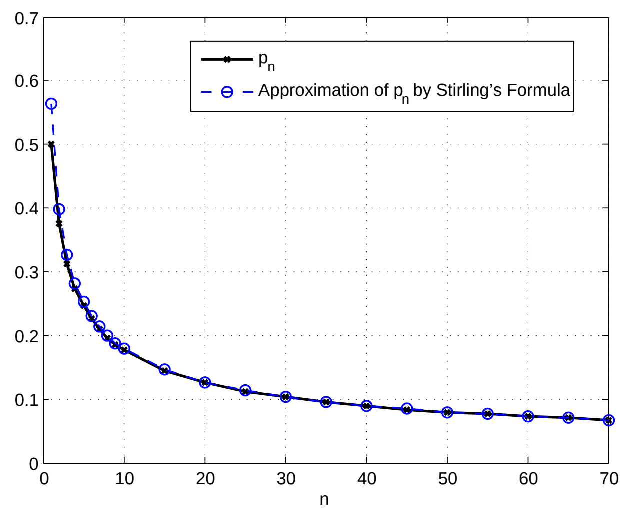

Figure 2.1: Comparison of p_n and its approximation via Stirling's formula