ECS 315: Probability and Random Processes **2019/1**

HW Solution 11 — Due: November 21, 4 PM

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Problem 1 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

(e) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & x > b. \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $F_X(x) = \boxed{\begin{cases} 0, & x < -5, \\ \frac{x+5}{10}, & a \le x \le b. \\ 1, & x > 5 \end{cases}}$

(c) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\,dx = \int\limits_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^2}{2}\right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}$.

In general,

$$\mathbb{E}X = \int\limits_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(d) $\mathbb{E}[X^5] = \int\limits_{-\infty}^{\infty} x^5 f_X(x)\,dx = \int\limits_{-5}^{5} x^5 \times \frac{1}{10}dx = \frac{1}{10}\left.\frac{x^6}{6}\right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}$.

In general,

$$\mathbb{E}\left[X^5\right] = \int\limits_a^b x^5 \frac{1}{b-a}dx = \frac{1}{b-a}\int\limits_a^b x^5 dx = \frac{1}{b-a}\left.\frac{x^6}{6}\right|_a^b = \frac{1}{b-a}\frac{b^6-a^6}{2}.$$

With $a=-5$ and $b=5$, we have $\mathbb{E}[X^5] = \boxed{0}$.

(e) In general,

$$\mathbb{E}\left[e^X\right] = \int\limits_a^b e^x \frac{1}{b-a}dx = \frac{1}{b-a}\int\limits_a^b e^x dx = \frac{1}{b-a}\left.e^x\right|_a^b = \frac{e^b-e^a}{b-a}.$$

With $a=-5$ and $b=5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5-e^{-5}}{10}} \approx 14.84$.

Problem 2 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta)f_\Theta(\theta)d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t+\Theta)] = 5\int_0^{2\pi}\frac{1}{2\pi}\cos(7t+\theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

(b) Y is another function of Θ.

$$\mathbb{E}[Y] = \mathbb{E}[5\cos(7t_1+\Theta)\times 5\cos(7t_2+\Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1+\theta)\times 5\cos(7t_2+\theta)d\theta$$
$$= \frac{25}{2\pi}\int_0^{2\pi} \cos(7t_1+\theta)\times\cos(7t_2+\theta)d\theta.$$

Recall[1] the cosine identity

$$\cos(a)\times\cos(b) = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$

Therefore,

$$\mathbb{E}Y = \frac{25}{4\pi}\int_0^{2\pi} \cos(7t_1+7t_2+2\theta)+\cos(7(t_1-t_2))\,d\theta$$
$$= \frac{25}{4\pi}\left(\int_0^{2\pi}\cos(7t_1+7t_2+2\theta)\,d\theta + \int_0^{2\pi}\cos(7(t_1-t_2))\,d\theta\right).$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1-t_2))\int_0^{2\pi} d\theta = \frac{25}{4\pi}\cos(7(t_1-t_2))\,2\pi = \boxed{\frac{25}{2}\cos(7(t_1-t_2))}.$$

[1]This identity could be derived easily via the Euler's identity:

$$\cos(a)\times\cos(b) = \frac{e^{ja}+e^{-ja}}{2}\times\frac{e^{jb}+e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb}+e^{-ja}e^{jb}+e^{ja}e^{-jb}+e^{-ja}e^{-jb}\right)$$
$$= \frac{1}{2}\left(\frac{e^{ja}e^{jb}+e^{-ja}e^{-jb}}{2}+\frac{e^{-ja}e^{jb}+e^{ja}e^{-jb}}{2}\right)$$
$$= \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$