ECS 315: In-Class Exercise # 11 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

1. Consider a random experiment in which you roll a six-sided fair dice (whose faces are numbered 1-6). We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega \quad \text{and} \quad Y(\omega) = 2 + \big((\omega-1)(\omega-3)(\omega-5)(\omega-7)\big).$$

a. Find $P[X = 2]$.

$X(\omega) = 2$ when $\omega = 2$. Therefore, $P[X = 2] = P(\{2\}) = \frac{1}{6}$.

b. Find $P[Y = 2]$.

$Y(\omega) = 2$ when $2 + \big((\omega-1)(\omega-3)(\omega-5)(\omega-7)\big) = 2$.

$\omega = 1,3,5,\cancel{7}$ Not in Ω.

Therefore, $P[Y = 2] = P(\{1,3,5\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

2. Consider a random experiment in which you roll a 10-sided fair dice (whose faces are numbered 0–9). Define a random variable Z from the outcomes of this experiment by

$$Z(\omega) = (\omega - 6)^2.$$

$\Omega = \{0,1,2,\dots,9\}$

a. Find $P[Z = 4]$.

$Z(\omega) = 4$ when $(\omega - 6)^2 = 4$.

$\omega = 6 \pm 2 = 4$ or 8.

Therefore, $P[Z = 4] = P(\{4,8\}) = \frac{1}{10} + \frac{1}{10} = \frac{2}{10}$.

b. Find $P[Z > 20]$.

Method 1:

$Z(\omega) > 20$ when $(\omega - 6)^2 > 20$.

$\cancel{\omega > 6 + \sqrt{20}}$ or $\omega < 6 - \sqrt{20}$

$\approx 10.4721 \qquad \approx 1.5279$

None of the ω in Ω satisfies this condition.

$\omega = 0$ or 1 Same

Therefore, $P[Z > 20] = P(\{0,1\}) = \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$.

Method 2: Because Ω is not large, it is possible to find $Z(\omega)$ for all ω.

ω	$\omega - 6$	$(\omega - 6)^2$
0	-6	36
1	-5	25
2	-4	16
3	-3	9
4	-2	4
5	-1	1
6	0	0
7	1	1
8	2	4
9	3	9

> 20

ECS 315: In-Class Exercise # 12 - Sol

Instructions

1. Separate into groups of no more than three students each.
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Name: Prapun

ID (last 3 digits): 5 5 5

Consider a random variable whose pmf is given by $p_X(x) = \begin{cases} \dfrac{1}{4}, & x = 1, 9, \\ c, & x = 5, \\ 0, & \text{otherwise.} \end{cases}$
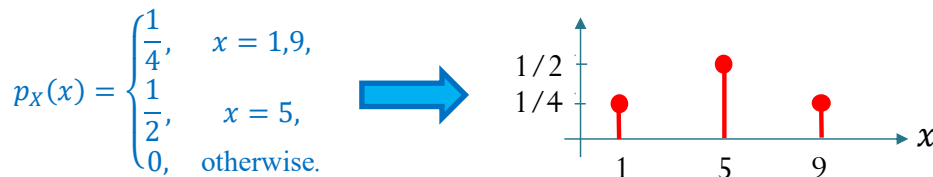
a) Find the constant c.

Recall that, for any pmf, $\sum_x p_X(x) = 1$. Therefore, we must have
$$p_X(1) + p_X(5) + p_X(9) = 1$$

$$\frac{1}{4} + c + \frac{1}{4} = 1$$
$$c = \frac{1}{2}.$$

b) Plot $p_X(x)$. (Recall that we use stem plot for pmf.)

$$p_X(x) = \begin{cases} \dfrac{1}{4}, & x = 1, 9, \\ \dfrac{1}{2}, & x = 5, \\ 0, & \text{otherwise.} \end{cases}$$

c) Find $P[X \leq 7]$.

$$P[X \leq 7] = p_X(1) + p_X(5) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

d) Find $P[X > 4]$.

$$P[X > 4] = p_X(5) + p_X(9) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

e) Find $P[X \leq 5]$.

$$P[X \leq 5] = p_X(1) + p_X(5) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

f) Find $P[X \leq 4.99]$.

$$P[X \leq 4.99] = p_X(1) = \frac{1}{4}.$$

g) Find $P[X \leq 5.01]$.

$$P[X \leq 5.01] = p_X(1) + p_X(5) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

ECS 315: In-Class Exercise # 13 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. [ENRE] Explanation is not required for this exercise.
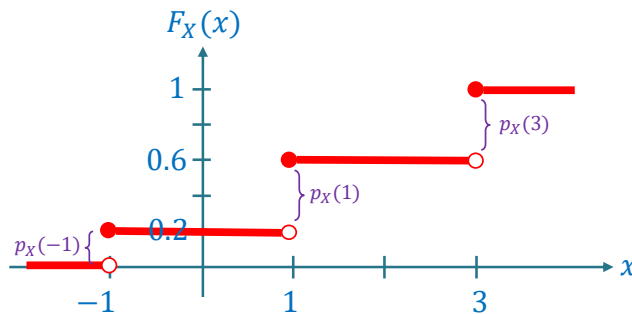3. **Do not panic.**

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1. Consider a random variable X whose pmf is given by

$$p_X(x) = \begin{cases} 0.2, & x = -1, \\ c, & x = 1, 3, \\ 0, & \text{otherwise.} \end{cases}$$

   a. Find the constant c.

   "$\Sigma = 1$" $\Rightarrow p_X(-1) + p_X(1) + p_X(3) = 1$
   $$0.2 + c + c = 1$$
   $$c = 0.4$$

   b. Plot the cdf of this random variable.

   Recall that the cdf can be derived from the pmf by using the $p_x(x)$ as the jump amount at x.

2. Consider a random variable X whose cdf is given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.2, & 0 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

   At $x = 0$, there is a jump of size 0.2.

   At $x = 3$, there is a jump of size 0.8.

   a. Find $P[X \le 1]$.

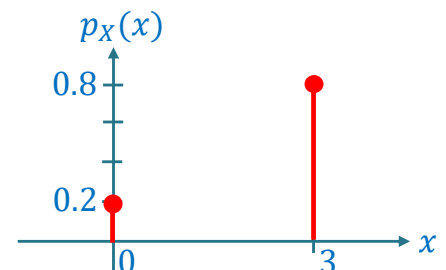   By definition, $P[X \le 1] = F_X(1) = 0.2$.

   b. Find $P[X > 1]$.

   Because $[X > 1]$ and $[X \le 1]$ are opposite (complementary) events, we know that
   $$P[X > 1] = 1 - P[X \le 1] = 1 - 0.2 = 0.8.$$

   c. Plot the pmf of X.

   For discrete RV, the pmf can be derived from the jump amounts in the cdf plot.
   Here, the jumps in the cdf happen twice: at $x = 0$ and $x = 3$.
   The jump amounts are 0.2 and 0.8, respectively.
   Therefore, $p_X(x) = \begin{cases} 0.2, & x = 0, \\ 0.8, & x = 2, 3 \\ 0, & \text{otherwise.} \end{cases}$

ECS 315: In-Class Exercise # 14 - Sol

Consider the random variable specified in each part below.

i) Write down its (minimal) support.

ii) Find $P[X = 0]$. Your answer should be of the form 0.XXXX.

iii) Find $P[X = 2]$. Your answer should be of the form 0.XXXX.

	(minimal) support	$P[X = 0]$	$P[X = 2]$
$X \sim \mathcal{U}(\{-2,0,2\})$ $X \sim \text{Uniform}(S)$	The minimal support of a uniform RV is the set S being specified. Here, $S = \{-2,0,2\}$.	$p_X(x) = \begin{cases} \frac{1}{\lvert S \rvert}, & x \in S, \\ 0, & \text{otherwise.} \end{cases}$ Here, $\lvert S \rvert = 3$. Because $0 \in S$, we have $p_X(0) = \frac{1}{3}$. Therefore, $$P[X = 0] = \frac{1}{3} \approx 0.3333.$$	Because $2 \in S$, we have $p_X(2) = \frac{1}{3}$. Therefore, $$P[X = 2] = \frac{1}{3} \approx 0.3333.$$
$X \sim \text{Bernoulli}\left(\frac{1}{4}\right)$ $X \sim \text{Bernoulli}(p)$	The (minimal) support of any Bernoulli RV is $\{0,1\}$.	$p_X(x) = \begin{cases} 1 - p, & x = 0, \\ p, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$ Here, $p = \frac{1}{4}$. $P[X = 0] = p_X(0) = 1 - p$ $= 1 - \frac{1}{4} = \frac{3}{4} = 0.7500.$	$P[X = 2] = p_X(2) = 0.0000.$
$X \sim \mathcal{B}(4, 0.6)$ $X \sim \text{Binomial}(n, p)$	The (minimal) support of a Binomial RV is $\{0, 1, \ldots, n\}$. Here, $n = 4$. Therefore, the (minimal) support is $\{0,1,2,3,4\}$.	$p_X(x) = \begin{cases} \binom{n}{x} p^x (1 - p)^{n-x}, & x = 0,1,2,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$ Here, $n = 4$ and $p = 0.6$. $p_X(x) = \begin{cases} \binom{4}{x} 0.6^x (1 - 0.6)^{4-x}, & x = 0,1,2,3,4, \\ 0, & \text{otherwise.} \end{cases}$ Therefore, $P[X = 0] = p_X(0) = \binom{4}{0} 0.6^0 (1 - 0.6)^4$ $= 0.4^4 = 0.0256$	$P[X = 2] = p_X(2)$ $= \binom{4}{2} 0.6^2 (1 - 0.6)^2$ $= 6 \times 0.6^2 \times 0.4^2$ $= 0.3456$

ECS 315: In-Class Exercise # 15 - Sol

Date: 2 2 / 1 0 / 2019

Name		ID (last 3 digits)		
Prapun		5	5	5

1. Suppose $X \sim \mathcal{G}_0\left(\frac{1}{3}\right)$. Find $P\left[X \le \frac{1}{3}\right]$.

   ➔ $p_X(x) = \begin{cases} p(1-p)^x, & x = 0,1,2,\dots, \\ 0, & \text{otherwise.} \end{cases}$

   The possible values of X are $0,1,2,\dots$. Among these, only 0 satisfies the condition "$\le \frac{1}{3}$".

   Therefore, $P\left[X \le \frac{1}{3}\right] = p_X(0) = p(1-p)^0 = p = \frac{1}{3}$.

2. Suppose $X \sim \mathcal{G}_1\left(\frac{1}{3}\right)$. Find $P\left[X \le \frac{1}{3}\right]$.

   ➔ $p_X(x) = \begin{cases} p(1-p)^{x-1}, & x = 1,2,\dots, \\ 0, & \text{otherwise.} \end{cases}$

   Note that the possible values (with positive probability) of X are always ≥ 1.
   Therefore, $P\left[X \le \frac{1}{3}\right] = 0$.

3. [ENRPa] Consider (a sequence of independent) Bernoulli trials whose success probability for each trial is $1/4$. For each of the random variables defined below, indicate the **name and the parameter(s)** of the family it belongs to.

Random Variable	Family
$K =$ the number of failures until the first success occurs.	$\mathcal{G}_0\left(\frac{1}{4}\right)$
$N =$ the number of successes among the first 4 trials.	$\mathcal{B}\left(4,\frac{1}{4}\right)$
$F =$ the number of successes until the first failure occurs.	$\mathcal{G}_0\left(\frac{3}{4}\right)$

   For K and N, see the interpretation part of Definitions 8.33 and 8.22. In addition, see Figure 22.

   An easy way to think about F is to note that F and K are almost the same except that the roles of "success" and "failure" are switched. So, it makes sense that it still belongs to the same family with the parameter being replaced by the opposite case.

   A formal way to solve this problem is to derive the pmf of F and compare the pmf to the known families. Here, because F is defined to be "the number of" something, we start by assuming that the possible values of F are $0,1,2,3,\dots$. $F = 0$ is possible because one may fail right at the first trial; this happens with probability $1 - \frac{1}{4} = \frac{3}{4}$. In general, for $k = 0,1,2,\dots$, $P[F = f] = \left(\frac{1}{4}\right)^f \times \frac{3}{4}$. By comparing the formula with the known families (Table 3 in the lecture notes), $P[F = f]$ fits the formula for the geometric$_0$ distribution; the corresponding p is $\frac{3}{4}$.

ECS 315: In-Class Exercise # 16 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

1. Arrivals of customers at a restaurant are modeled by a Poisson process with a rate of $\lambda = \frac{1}{3}$ customers per minute. Let X be the number of customers arriving between 5:00 PM and 5:06 PM.

   a. Plot its cdf $F_X(x)$ on the interval $[-3, 3]$.
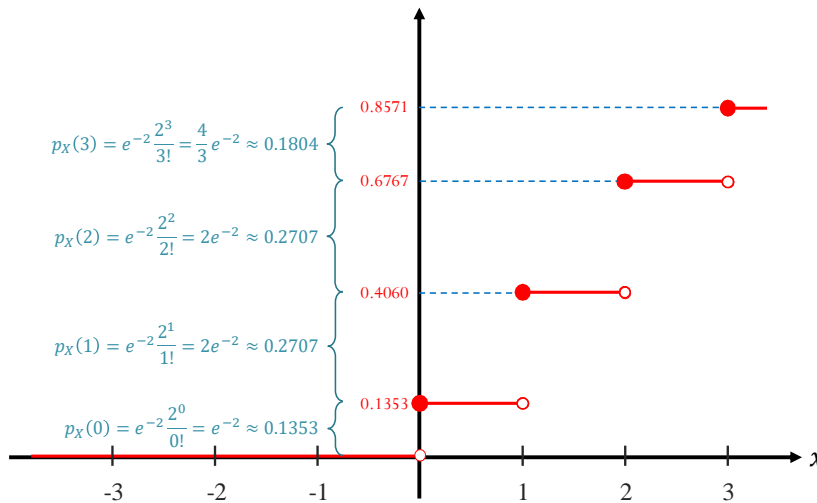
We know that $X \sim \mathcal{P}(\alpha)$ where $\alpha = \lambda T$. Here,

$\lambda = \frac{1}{3}$ [customers per minute] and $T = 6$ [minutes].

Therefore,

$$\alpha = \frac{1}{3} \times 6 = 2.$$

Being a Poisson RV, we also know that

$$p_X(x) = \begin{cases} e^{-\alpha} \dfrac{\alpha^x}{x!}, & x = 0,1,2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

$p_X(3) = e^{-2}\frac{2^3}{3!} = \frac{4}{3}e^{-2} \approx 0.1804$

$p_X(2) = e^{-2}\frac{2^2}{2!} = 2e^{-2} \approx 0.2707$

$p_X(1) = e^{-2}\frac{2^1}{1!} = 2e^{-2} \approx 0.2707$

$p_X(0) = e^{-2}\frac{2^0}{0!} = e^{-2} \approx 0.1353$

   b. What is the probability that $X < 1$?

   Because $X \sim \mathcal{P}(\alpha)$, the possible values of X are $0,1,2,3,\ldots$ Among its possible values, the only value that is "< 1" is "0". Therefore,

$$P[X < 1] = P[X = 0] = p_X(0) = e^{-2}\frac{2^0}{0!} = e^{-2} \approx 0.1353.$$

2. Let N be the number of successes in 10^{20} Bernoulli trials. Assume that the probability of success for each trial is 10^{-21}. Use Poisson approximation to calculate $P[N = 0]$.

   Your answer should be of the form 0.XXXX.
   First, note that $N \sim \mathcal{B}(n, p)$ where $n = 10^{20}$ and $p = 10^{-21}$. Because n is large and p is small, we can approximate the pmf of N by that of the Poisson pmf whose $\alpha = np$. Here, $np = 10^{20} \times 10^{-21} = \frac{1}{10}$. Therefore,

$$P[N = n] \approx \frac{e^{-\alpha}\alpha^n}{n!} \text{ for } n = 0,1,2, \ldots$$

   Substituting $\alpha = 10$, we have

$$P[N = 0] \approx \frac{e^{-\frac{1}{10}}10^0}{0!} = e^{-\frac{1}{10}} \approx 0.9048.$$

ECS 315: In-Class Exercise # 17 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

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1. Find the expected value of the random variable X defined in each part below:

a. $p_X(x) = \begin{cases} cx, & x \in \{1,2\}, \\ 0, & \text{otherwise.} \end{cases}$

First, we need to solve for the value of the unknown constant c.

To be a pmf, we need "$\Sigma = 1$". So,

$$p_X(1) + p_X(2) = 1$$
$$c + 2c = 1$$
$$c = \frac{1}{3}.$$

x	$p_X(x)$
1	$c = \frac{1}{3}$
2	$2c = 2 \times \frac{1}{3} = \frac{2}{3}$

$$\mathbb{E}X = \sum_x x p_X(x) = \left(1 \times p_X(1)\right) + \left(2 \times p_X(2)\right)$$

$$= \left(1 \times \frac{1}{3}\right) + \left(2 \times \frac{2}{3}\right) = \frac{5}{3} \approx 1.67.$$

b. $p_X(x) = \begin{cases} 0.3, & x = -1, 1, \\ c, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$

"$\Sigma = 1$": $\quad p_X(-1) + p_X(1) + p_X(3) = 1$

$$0.3 + 0.3 + c = 1$$
$$c = 0.4.$$

x	$p_X(x)$
-1	0.3
1	0.3
3	$c = 0.4$

$\mathbb{E}X = \sum_x x p_X(x) = (-1 \times 0.3) + (1 \times 0.3) + (3 \times 0.4) = 1.2.$

c. $F_X(x) = \begin{cases} 0, & x < -1, \\ 0.4, & -1 \le x < 1, \\ 1, & x \ge 1. \end{cases}$

This cdf has two jumps; one is @ $x = -1$ and another one is @ $x = 1$.

The jump sizes are 0.4 and 0.6, respectively.

x	$p_X(x)$
-1	0.4
1	0.6

$\mathbb{E}X = \sum_x x p_X(x) = (-1 \times 0.4) + (1 \times 0.6) = 0.2.$

d. $p_X(x) = \begin{cases} cx, & x \in \{1,2,3,\ldots,10\}, \\ 0, & \text{otherwise.} \end{cases}$

"$\Sigma = 1$": $\sum_{x=1}^{10} cx = 1 \Rightarrow c = \frac{1}{\sum_{x=1}^{10} x} = \frac{2}{10 \times 11}.$

$\mathbb{E}X = \sum_x x p_X(x) = \sum_{x=1}^{10} x(cx) = c \sum_{x=1}^{10} x^2 = c\left(\frac{1}{6} \times 10 \times 11 \times 21\right) = 7.$

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2 + n}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^4 + 2n^3 + n^2}{4}$$

ECS 315: In-Class Exercise # 18

Date: 2 9 / 1 0 / 2019

Name	ID (last 3 digits)		
Prapun	5	5	5

1. Find $\mathbb{E}\left[X^2\right]$ and $\mathrm{Var}\left[X\right]$ for the random variable X defined in each part below:

$p_X(x)$	$\mathbb{E}[X]$	$\mathbb{E}\left[X^2\right]$	$\mathrm{Var}[X]$
$\begin{cases} \dfrac{1}{3}x, & x \in \{1,2\}, \\ 0, & \text{otherwise.} \end{cases}$	$\dfrac{5}{3}$	$\begin{aligned} &= \sum_x x^2 p_X(x) \\ &= \left(1^2 \times p_X(1)\right) + \left(2^2 \times p_X(2)\right) \\ &= \left(1 \times \dfrac{1}{3}\right) + \left(4 \times \dfrac{2}{3}\right) = \dfrac{9}{3} = 3. \end{aligned}$	$\begin{aligned} &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 \\ &= \dfrac{9}{3} - \left(\dfrac{5}{3}\right)^2 = \dfrac{2}{9} \end{aligned}$
$\begin{cases} 0.3, & x = -1,1, \\ 0.4, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$	1.2	$\begin{aligned} &= \sum_x x^2 p_X(x) \\ &= \left((-1)^2 \times p_X(-1)\right) + \left(1^2 \times p_X(1)\right) + \left(3^2 \times p_X(3)\right) \\ &= (1 \times 0.3) + (1 \times 0.3) + (9 \times 0.4) \\ &= 4.2 = \dfrac{21}{5} \end{aligned}$	$\begin{aligned} &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 \\ &= 4.2 - (1.2)^2 \\ &= 2.76 = \dfrac{69}{25} \end{aligned}$
$\begin{cases} \dfrac{1}{55}x, & x \in \{1,2,3,\ldots,10\}, \\ 0, & \text{otherwise.} \end{cases}$	7	$\begin{aligned} &= \sum_x x^2 p_X(x) = \sum_{x=1}^{10} x^2 \left(\dfrac{1}{55}x\right) \\ &= \dfrac{1}{55} \sum_{x=1}^{10} x^3 = \dfrac{55^2}{55} = 55 \end{aligned}$ Note that $$\sum_{x=1}^{n} x^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$ $$= \left[\dfrac{n(n+1)}{2}\right]^2$$	$\begin{aligned} &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 \\ &= 55 - (7)^2 \\ &= 6 \end{aligned}$

ECS 315: In-Class Exercise # 19

Date: 0 5 / 1 1 / 2019		
Name		ID (last 3 digits)
Prapun		5 5 5

In this question, we consider two distributions for a random variable X.

In part (a), which corresponds to the second column in the table below, X is a **discrete** random variable with its pmf specified in the first row.

In part (b), which corresponds to the third column, X is a **continuous** random variable with its pdf specified in the first row.

	$p_X(x) = \begin{cases} cx^2, & x \in \{-1,2\}, \\ 0, & \text{otherwise.} \end{cases}$	$f_X(x) = \begin{cases} cx^2, & x \in (-1,2], \\ 0, & \text{otherwise.} \end{cases}$
Find c	"$\Sigma = 1$": $p_X(-1) + p_X(2) = 1$ $c(-1)^2 + c(2)^2 = 1$ $5c = 1$ $c = \dfrac{1}{5}.$	We need "$\int = 1$". $\displaystyle\int_{-\infty}^{\infty} f_X(x)dx = \int_{-1}^{2} f_X(x)dx$ $\displaystyle = \int_{-1}^{2} cx^2 dx = \left. \frac{cx^3}{3} \right|_{-1}^{2}$ $\displaystyle = \frac{c}{3}(8-(-1)) = 3c$ Therefore, $3c = 1$ $c = \dfrac{1}{3}.$
Find $P[X=2]$	$P[X = 2] = p_X(2) = c(2)^2 = \dfrac{4}{5}.$	Method 1: X is a continuous RV. For any continuous RV, $P[X = x] = 0$ for any x. Therefore, $P[X = 2] = 0$. Method 2: $\displaystyle P[X = 2] = \int_2^2 f_X(x)dx = \int_2^2 cx^2 dx$ $\displaystyle = \left. \frac{cx^3}{3} \right|_2^2 = 0.$
Find $P[0 < X \leq 2]$	The possible values of this RV are -1 and 2. Among these, only "2" satisfies the condition. Therefore, $P[0 < X \leq 2] = P[X = 2] = \dfrac{4}{5}.$	$\displaystyle P[0 < X \leq 2] = \int_0^2 f_X(x)dx = \int_0^2 cx^2 dx$ $\displaystyle = \left. \frac{cx^3}{3} \right|_0^2 = \frac{8}{3}c = \frac{8}{9}.$

ECS 315: In-Class Exercise # 20

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. [ENRE] Explanation is not required for this exercise..
3. **Do not panic.**

Date: 0 7 / 1 1 / 2019

Name		ID (last 3 digits)		
Prapun		5	5	5

In this question, we consider two distributions for a random variable X.

In part (a), which corresponds to the second column in the table below, X is a **discrete** random variable with its pmf specified in the first row.

In part (b), which corresponds to the third column, X is a **continuous** random variable with its pdf specified in the first row.

	$$p_X(x) = \begin{cases} \dfrac{1}{5}x^2, & x \in \{-1, 2\}, \\ 0, & \text{otherwise.} \end{cases}$$	$$f_X(x) = \begin{cases} \dfrac{1}{3}x^2, & x \in (-1, 2], \\ 0, & \text{otherwise.} \end{cases}$$
Find the cdf $F_X(x)$	For discrete RV, it may be easier to work on the plot of cdf first, then come back here. $$F_X(x) = \begin{cases} 0, & x < -1, \\ 0.2, & -1 \le x < 2, \\ 1 & x \ge 2. \end{cases}$$ Note that, for discrete RVs, their cdf's are not continuous. At all points, including the jumps, the cdf's must be right-continuous. Therefore, here, the interval that $F_X(x) = 0.2$ must include $x = -1$ and exclude $x = 2$.	$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$ For $x \le -1$, $F_X(x) = \int_{-\infty}^{x} 0 dt = 0$. For $-1 < x \le 2$, $$F_X(x) = \int_{-\infty}^{-1} 0 dt + \int_{-1}^{x} \frac{1}{3}t^2 dt$$ $$= 0 + \frac{t^3}{9}\Big|_{-1}^{x} = \frac{x^3 + 1}{9}.$$ For $x > 2$, $F_X(x) = \int_{-\infty}^{2} \frac{1}{3}t^2 dt + \int_{2}^{\infty} 0 dt = 1$. Therefore, $$F_X(x) = \begin{cases} 0, & x \le -1, \\ \dfrac{x^3 + 1}{9}, & -1 < x \le 2, \\ 1 & x > 2. \end{cases}$$
Plot the cdf $F_X(x)$	To get the cdf from the pmf, locate where the probability masses are from the pmf. Then, start from $-\infty$ with the value 0. Moving to the right, we increase the cdf value at the locations of the probability masses. The amount of jump is the same as the amount of probability mass at that location.	The expression in the previous part gives

ECS 315: In-Class Exercise # 21

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Name	ID (last 3 digits)		
Prapun	5	5	5

In this question, we consider two distributions for a random variable X.

In part (a), which corresponds to the second column in the table below, X is a **discrete** random variable with its pmf specified in the first row.

In part (b), which corresponds to the third column, X is a **continuous** random variable with its pdf specified in the first row.

	$p_X(x) = \begin{cases} \dfrac{1}{5}x^2, & x \in \{-1,2\}, \\ 0, & \text{otherwise.} \end{cases}$	$f_X(x) = \begin{cases} \dfrac{1}{3}x^2, & x \in (-1,2], \\ 0, & \text{otherwise.} \end{cases}$
Find $\mathbb{E}X$	$\mathbb{E}X = \sum_x x p_X(x) = (-1)\left(\dfrac{1}{5}\right) + (2)\left(\dfrac{4}{5}\right)$ $= \dfrac{7}{5} = 1.4$	$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-1}^{2} x\left(\dfrac{1}{3}x^2\right)dx$ $= \dfrac{1}{3}\dfrac{x^4}{4}\Big|_{-1}^{2} = \dfrac{1}{12}(16-1) = \dfrac{15}{12} = \dfrac{5}{4}$ $= 1.25$
Find $\mathbb{E}\left[X^2\right]$	$\mathbb{E}[X^2] = \sum_x x^2 p_X(x)$ $= (-1)^2\left(\dfrac{1}{5}\right) + (2)^2\left(\dfrac{4}{5}\right)$ $= \dfrac{17}{5} = 3.4$	$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx$ $= \int_{-1}^{2} x^2\left(\dfrac{1}{3}x^2\right)dx = \dfrac{1}{3}\int_{-1}^{2} x^4 dx$ $= \dfrac{1}{3}\dfrac{x^5}{5}\Big|_{-1}^{2} = \dfrac{1}{15}(32-(-1)) = \dfrac{33}{15}$ $= \dfrac{11}{5} = 2.2$
Find $\text{Var}\left[X\right]$	$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$ $= \dfrac{17}{5} - \left(\dfrac{7}{5}\right)^2 = \dfrac{85-49}{25}$ $= \dfrac{36}{25} = 1.44$	$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$ $= \dfrac{11}{5} - \left(\dfrac{5}{4}\right)^2 = \dfrac{176-125}{80} = \dfrac{51}{80}$ ≈ 0.6375
Find σ_X	$\sigma_X = \sqrt{\text{Var}[X]} = \sqrt{\dfrac{36}{25}} = \dfrac{6}{5} = 1.2$	$\sigma_X = \sqrt{\text{Var}[X]} = \sqrt{\dfrac{51}{80}} = \dfrac{1}{4}\sqrt{\dfrac{51}{5}} \approx 0.7984$

ECS 315: In-Class Exercise # 22 - Sol

Instructions

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Date: 1 4 / 1 1 / 2019		
Name	ID (last 3 digits)	
Prapun	5	5	5

Calculate $P[0.5 < X \leq 1.5]$ for each of the following random variables.

Your answer should be of the form 0.XXXX.

a) $X \sim \mathcal{U}(\overset{a}{1}, \overset{b}{4})$

$$f_X(X) = \begin{cases} \dfrac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \dfrac{1}{3}, & 1 < x < 4, \\ 0, & \text{otherwise.} \end{cases}$$

$$P[0.5 < X < 1.5] = \int_{0.5}^{1.5} f_X(x)dx$$
$$= \int_{0.5}^{1} 0 \, dx + \int_{1}^{1.5} \dfrac{1}{3} dx$$
$$= 0 + \dfrac{1}{3}x \Big|_{x=1}^{1.5} = \dfrac{1}{6}$$
$$\approx 0.1667$$

b) $X \sim \mathcal{E}(\overset{\lambda}{1})$

$$f_X(X) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$P[0.5 < X < 1.5] = \int_{0.5}^{1.5} f_X(x)dx$$
$$= \int_{0.5}^{1.5} e^{-x}dx = -e^{-x}\Big|_{x=0.5}^{1.5}$$
$$= e^{-0.5} - e^{-1.5}$$
$$\approx 0.6065 - 0.2231$$
$$= 0.3834$$

c) $X \sim \mathcal{N}(\overset{m, \sigma^2}{0, 1})$

$$P[0.5 < X < 1.5] = F_X(1.5) - F_X(0.5) = \Phi(1.5) - \Phi(0.5)$$
$$\approx 0.9332 - 0.6915 = 0.2417$$

We can use the ϕ-function directly here because X is a standard Gaussian RV.

d) $X \sim \mathcal{N}(\overset{m, \sigma^2}{1, 3})$

$$P[0.5 < X < 1.5] = F_X(1.5) - F_X(0.5) = \Phi\left(\dfrac{1.5-1}{\sqrt{3}}\right) - \Phi\left(\dfrac{0.5-1}{\sqrt{3}}\right)$$
$$\approx \Phi(0.29) - \Phi(-0.29) = \Phi(0.29) - (1 - \Phi(0.29)) \approx 2\Phi(0.29) - 1$$
$$= 2 \times 0.6141 - 1 = 0.2282$$

ECS 315: In-Class Exercise # 23 - Sol

Instructions

Name	ID (last 3 digits)		
Prapun	5	5	5

1. Separate into groups of no more than three students each. **The group cannot be the same as any of your former groups after the midterm.**
2. Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.
3. **Do not panic.**

Random variables X and Y have the following joint pmf

$$p_{X,Y}(x,y) = \begin{cases} c(x+y)^2, & x \in \{-1,2\} \text{ and } y \in \{0,-3\}, \\ 0, & \text{otherwise.} \end{cases}$$

a) Find c

$$x+y \quad \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & -1 & -4 \\ 2 & 2 & -1 \end{array} \implies c(x+y)^2 \quad \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & c & 16c \\ 2 & 4c & c \end{array}$$

$$\sum_x \sum_y p_{X,Y}(x,y) = c + 16c + 4c + c = 22c$$

$$1 = 22c$$

$$c = \frac{1}{22}$$

b) Find the joint pmf matrix $\mathbf{P}_{X,Y}$

$$p_{X,Y}(x,y) = c(x+y)^2 \quad \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & c & 16c \\ 2 & 4c & c \end{array} \implies \mathbf{P}_{X,Y} \quad \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & \frac{1}{22} & \frac{16}{22} \\ 2 & \frac{4}{22} & \frac{1}{22} \end{array} = \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & \frac{1}{22} & \frac{8}{11} \\ 2 & \frac{2}{11} & \frac{1}{22} \end{array}$$

c) Find $P[X > Y]$.

We circle only the elements whose $x > y$.

$$\mathbf{P}_{X,Y} \quad \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & \frac{1}{22} & \frac{16}{22} \\ 2 & \frac{4}{22} & \frac{1}{22} \end{array} \implies P[X > Y] = \frac{16 + 4 + 1}{22} = \frac{21}{22} \approx 0.9545$$

d) Find the pmf $p_X(x)$ and the pmf $p_Y(y)$.

$$\mathbf{P}_{X,Y} \quad \begin{array}{c|cc} x \backslash y & 0 & -3 \\ \hline -1 & \frac{1}{22} & \frac{16}{22} \\ 2 & \frac{4}{22} & \frac{1}{22} \end{array} \xrightarrow{\sum} \begin{array}{c} \frac{17}{22} \\ \frac{5}{22} \end{array}$$

$$\downarrow \sum \quad \downarrow \sum$$

$$\frac{5}{22} \quad \frac{17}{22}$$

$$p_Y(y) = \begin{cases} \frac{5}{22}, & y = 0, \\ \frac{17}{22}, & y = -3, \\ 0, & \text{otherwise} \end{cases}$$

$$p_X(x) = \begin{cases} \frac{17}{22}, & x = -1, \\ \frac{5}{22}, & x = 2, \\ 0, & \text{otherwise} \end{cases}$$

1) Suppose the pmf of a random variable X is given by

$$p_X(x) = \begin{cases} 0.1, & x = 2, \\ c, & x = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Let Y be another random variable. Assume that X and Y are i.i.d.

Find

a) $c =$ __0.9__

pmf $\Rightarrow$ "$\Sigma = 1$" $\Rightarrow$ $0.1 + c = 1$ $\Rightarrow$ $c = 0.9$ $\Rightarrow$ $p_X(x) = \begin{cases} 0.1, & x = 2, \\ 0.9, & x = 4, \\ 0, & \text{otherwise.} \end{cases}$

X and Y are identically distributed.

b) Their joint pmf matrix $\mathbf{P}_{X,Y}$.

X and Y are independent.

$p_Y(y) = \begin{cases} 0.1, & y = 2, \\ 0.9, & y = 4, \\ 0, & \text{otherwise.} \end{cases}$

$p_{X,Y}(x, y) = p_X(x)p_Y(y)$

$$\mathbf{P}_{X,Y} = \begin{array}{c} x \backslash y \\ 2 \\ 4 \end{array} \begin{bmatrix} p_X(2)p_Y(2) & p_X(2)p_Y(4) \\ p_X(4)p_Y(2) & p_X(4)p_Y(4) \end{bmatrix} = \begin{bmatrix} (0.1)(0.1) & (0.1)(0.9) \\ (0.9)(0.1) & (0.9)(0.9) \end{bmatrix} = \begin{array}{c} x \backslash y \\ 2 \\ 4 \end{array} \begin{bmatrix} 0.01 & 0.09 \\ 0.09 & 0.81 \end{bmatrix}$$

2) Random variables X and Y are independent. Their joint pmf matrix is

$$\mathbf{P}_{X,Y} = \begin{array}{c} x \backslash y \\ -1 \\ 3 \end{array} \begin{bmatrix} 0.08 & 0.12 & a \\ 0.12 & b & c \end{bmatrix}.$$

Find the values of the unknown constants:

$a =$ __0.2__, $b =$ __0.18__, $c =$ __0.3__.

$p_{X,Y}(-1,2) = p_X(-1)p_Y(2)$

$0.08 = (0.08 + 0.12 + a)(0.08 + 0.12)$

$a = 0.2$

"$\Sigma = 1$"

$1 = 0.08 + 0.12 + 0.2 + 0.12 + 0.18 + c$

$c = 0.3$

$p_{X,Y}(-1,5) = p_X(-1)p_Y(5)$

$0.12 = (0.08 + 0.12 + 0.2)(0.12 + b)$

$b = 0.18$

ECS 315: In-Class Exercise # 25 - Sol

1. Random variables X and Y have the following joint pmf matrix

$$\mathbf{P}_{X,Y} = \begin{array}{c} x \backslash y \\ 1 \\ 0 \end{array} \begin{array}{cc} 0 & 1 \\ \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & c \end{bmatrix} \end{array}$$

Are X and Y uncorrelated?

$c = 1 - (0.1 + 0.2 + 0.3) = 1 - 0.6 = 0.4.$

$\mathbb{E}[XY] = 0 \times 1 \times 0.1 + 0 \times 0 \times 0.2 + 1 \times 1 \times 0.3 + 0 \times 1 \times 0.4 = 0.3$

$$\mathbf{P}_{X,Y} \quad \begin{array}{c} x \backslash y \\ 1 \\ 0 \end{array} \begin{array}{cc} 0 & 1 \\ \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \end{array} \begin{array}{c} \xrightarrow{\Sigma} 0.4 \\ \xrightarrow{\Sigma} 0.6 \end{array}$$

$\mathbb{E}X = 1 \times 0.4 + 0 \times 0.6 = 0.4$

$(\mathbb{E}X)(\mathbb{E}Y) = 0.28$

$\downarrow\Sigma \quad \downarrow\Sigma$

$0.3 \quad 0.7$

$\mathbb{E}Y = 0 \times 0.3 + 1 \times 0.7 = 0.7$

$(\mathbb{E}X)(\mathbb{E}Y) \neq \mathbb{E}[XY] \Rightarrow \text{Cov}[X,Y] \neq 0 \Rightarrow$ No, X and Y are not uncorrelated.

2. Random variables X and Y have the following joint pmf matrix

$$\mathbf{P}_{X,Y} = \begin{array}{c} x \backslash y \\ 1 \\ 0 \end{array} \begin{array}{cc} 0 & 1 \\ \begin{bmatrix} 0.1 & a \\ 0.2 & b \end{bmatrix} \end{array}$$

Suppose X and Y are uncorrelated. Find the values of the unknown constants:

$a = \underline{\frac{7}{30} \approx 0.2333}, \quad b = \underline{\frac{7}{15} \approx 0.4667}.$

$$\mathbf{P}_{X,Y} \quad \begin{array}{c} x \backslash y \\ 1 \\ 0 \end{array} \begin{array}{cc} 0 & 1 \\ \begin{bmatrix} 0.1 & a \\ 0.2 & b \end{bmatrix} \end{array} \begin{array}{c} \xrightarrow{\Sigma} 0.1 + a \\ \xrightarrow{\Sigma} 0.2 + b \end{array}$$

$\downarrow\Sigma \quad \downarrow\Sigma$

$\begin{array}{cc} 0.3 & a+b \\ & = 1 - 0.3 \\ & = 0.7 \end{array}$

$\mathbb{E}[XY] = (0)(1)(0.1) + (0)(0)(0.2) + (1)(1)(a) + (0)(1)(b)$
$\quad = a$

$\mathbb{E}X = 1 \times (0.1 + a) + 0 \times (0.2 + b) = 0.1 + a$

$\mathbb{E}Y = 0 \times (0.3) + 1 \times (0.7) = 0.7$

For X and Y to be uncorrelated, we need

$\mathbb{E}[XY] = (\mathbb{E}X)(\mathbb{E}Y)$

$a = (0.1 + a)(0.7)$

$a = \frac{7}{30} \approx 0.2333.$

$b = 0.7 - \frac{7}{30} = \frac{7}{15} \approx 0.4667.$