ECS 315: Probability and Random Processes **2018/1**

HW Solution 9 — Due: Nov 13, 4 PM

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Problem 1. Consider a random variable X whose pmf is

$$p_X(x) = \begin{cases} 1/2, & x = -1, \\ 1/4, & x = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2$.

(a) Find $\mathbb{E}X$.

(b) Find $\mathbb{E}[X^2]$.

(c) Find $\operatorname{Var} X$.

(d) Find σ_X.

(e) Find $p_Y(y)$.

(f) Find $\mathbb{E}Y$.

(g) Find $\mathbb{E}[Y^2]$.

Solution:

(a) $\mathbb{E}X = \sum_x x p_X(x) = (-1) \times \frac{1}{2} + (0) \times \frac{1}{4} + (1) \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = \boxed{-\frac{1}{4}}$.

(b) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = (-1)^2 \times \frac{1}{2} + (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$.

(c) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{3}{4} - \left(-\frac{1}{4}\right)^2 = \frac{3}{4} - \frac{1}{16} = \boxed{\frac{11}{16}}$.

(d) $\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\frac{\sqrt{11}}{4}}$.

(e) First, we build a table to see which values y of Y are possible from the values x of X:

x	$p_X(x)$	y
-1	1/2	$(-1)^2 = 1$
0	1/4	$(0)^2 = 0$
1	1/4	$(1)^2 = 1$

Therefore, the random variable Y can takes two values: 0 and 1. $p_Y(0) = p_X(0) = 1/4$. $p_Y(1) = p_X(-1) + p_X(1) = 1/2 + 1/4 = 3/4$. Therefore,

$$\boxed{p_Y(y) = \begin{cases} 1/4, & y = 0, \\ 3/4, & y = 1, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) $\mathbb{E}Y = \sum_y y p_Y(y) = (0) \times \frac{1}{4} + (1) \times \frac{3}{4} = \boxed{\frac{3}{4}}$. Alternatively, because $Y = X^2$, we automatically have $\mathbb{E}[Y] = \mathbb{E}[X^2]$. Therefore, we can simply use the answer from part (b).

(g) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{3}{4} = \boxed{\frac{3}{4}}$. Alternatively,

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[X^4\right] = \sum_x x^4 p_X(x) = (-1)^4 \times \frac{1}{2} + (0)^4 \times \frac{1}{4} + (1)^4 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Problem 2. For each of the following random variables, find $\mathbb{E}X$ and σ_X.

(a) $X \sim \text{Binomial}(3, 1/3)$

(b) $X \sim \text{Poisson}(3)$

Solution:

(a) From the lecture notes, we know that when $X \sim \text{Binomial}(n, p)$, we have $\mathbb{E}X = np$ and $\operatorname{Var} X = np(1-p)$. Here, $n = 3$ and $p = 1/3$. Therefore, $\mathbb{E}X = 3 \times \frac{1}{3} = \boxed{1}$. Also, because $\operatorname{Var} X = 3\left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right) = \frac{2}{3}$, we have $\sigma_X = \sqrt{\operatorname{Var} X} = \boxed{\sqrt{\frac{2}{3}}}$.

(b) From the lecture notes, we know that when $X \sim \text{Poisson}(\alpha)$, we have $\mathbb{E}X = \alpha$ and $\operatorname{Var} X = \alpha$. Here, $\alpha = 3$. Therefore, $\mathbb{E}X = \boxed{3}$. Also, because $\operatorname{Var} X = 3$, we have $\sigma_X = \boxed{\sqrt{3}}$.

Problem 3. Suppose X is a uniform discrete random variable on $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Find

(a) $\mathbb{E}X$

(b) $\mathbb{E}[X^2]$

(c) $\operatorname{Var} X$

(d) σ_X

Solution: All of the calculations in this question are simply plugging in numbers into appropriate formulas.

(a) $\mathbb{E}X = \boxed{0.5}$

(b) $\mathbb{E}[X^2] = \boxed{5.5}$

(c) $\operatorname{Var} X = \boxed{5.25}$

(d) $\sigma_X = \boxed{2.2913}$

Alternatively, we can find a formula for the general case of uniform random variable X on the sets of integers from a to b. Note that there are $n = b - a + 1$ values that the random variable can take. Hence, all of them has probability $\frac{1}{n}$.

(a) $\mathbb{E}X = \sum_{k=a}^{b} k\frac{1}{n} = \frac{1}{n}\sum_{k=a}^{b} k = \frac{1}{n} \times \frac{n(a+b)}{2} = \frac{a+b}{2}$.

(b) First, note that

$$\begin{aligned}\sum_{i=a}^{b} k\,(k-1) &= \sum_{k=a}^{b} k\,(k-1)\left(\frac{(k+1)-(k-2)}{3}\right)\\ &= \frac{1}{3}\left(\sum_{k=a}^{b} (k+1)\,k\,(k-1) - \sum_{k=a}^{b} k\,(k-1)\,(k-2)\right)\\ &= \frac{1}{3}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right)\end{aligned}$$

where the last equality comes from the fact that there are many terms in the first sum that is repeated in the second sum and hence many cancellations.

Now,

$$\begin{aligned}\sum_{k=a}^{b} k^2 &= \sum_{k=a}^{b} \left(k\,(k-1) + k\right) = \sum_{k=a}^{b} k\,(k-1) + \sum_{k=a}^{b} k\\ &= \frac{1}{3}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right) + \frac{n\,(a+b)}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=a}^{b} k^2 \frac{1}{n} &= \frac{1}{3n}\left((b+1)\,b\,(b-1) - a\,(a-1)\,(a-2)\right) + \frac{a+b}{2}\\ &= \frac{1}{3}\,a^2 - \frac{1}{6}\,a + \frac{1}{3}\,ab + \frac{1}{6}\,b + \frac{1}{3}\,b^2\end{aligned}$$

(c) $\operatorname{Var} X = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 = \frac{1}{12}\,(b-a)\,(b-a+2) = \frac{1}{12}(n-1)(n+1) = \frac{n^2-1}{12}$.

(d) $\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{\frac{n^2-1}{12}}$.

Problem 4. (Expectation + pmf + Gambling + Effect of miscalculation of probability) In the eighteenth century, a famous French mathematician Jean Le Rond d'Alembert, author of several works on probability, analyzed the toss of two coins. He reasoned that because this experiment has THREE outcomes, (the number of heads that turns up in those two tosses can be 0, 1, or 2), the chances of each must be 1 in 3. In other words, if we let N be the number of heads that shows up, Alembert would say that

$$p_N(n) = 1/3 \quad \text{for} \quad N = 0, 1, 2. \tag{9.1}$$

[Mlodinow, 2008, p 50–51]

We know that Alembert's conclusion was ***wrong***. His three outcomes are not equally likely and hence classical probability formula can not be applied directly. The key is to realize that there are FOUR outcomes which are equally likely. We should not consider 0, 1, or 2 heads as the possible outcomes. There are in fact four equally likely outcomes: (heads, heads), (heads, tails), (tails, heads), and (tails, tails). These are the 4 possibilities that make up the sample space. The actual pmf for N is

$$p_N(n) = \begin{cases} 1/4, & n = 0, 2, \\ 1/2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose you travel back in time and meet Alembert. You could make the following bet with Alembert to gain some easy money. The bet is that if the result of a toss of two coins contains exactly one head, then he would pay you \$150. Otherwise, you would pay him \$100.

Let R be Alembert's profit from this bet and Y be the your profit from this bet.

(a) Then, $R = -150$ if you win and $R = +100$ otherwise. Use Alembert's *miscalculated* probabilities from (9.1) to determine the pmf of R (from Alembert's belief).

(b) Use Alembert's *miscalculated* probabilities from (9.1) (or the corresponding (miscalculated) pmf found in part (a)) to calculate $\mathbb{E}R$, the expected profit for Alembert.

Remark: You should find that $\mathbb{E}R > 0$ and hence Alembert will be quite happy to accept your bet.

(c) Use the *actual* probabilities, to determine the pmf of R.

(d) Use the *actual* pmf, to determine $\mathbb{E}R$.

Remark: You should find that $\mathbb{E}R < 0$ and hence Alembert should not accept your bet if he calculates the probabilities correctly.

(e) Note that $Y = +150$ if you win and $Y = -100$ otherwise. Use the *actual* probabilities to determine the pmf of Y.

(f) Use the *actual* probabilities, to determine $\mathbb{E}Y$.

Remark: You should find that $\mathbb{E}Y > 0$. This is the amount of money that you expect to gain each time that you play with Alembert. Of course, Alembert, who still believes that his calculation is correct, will ask you to play this bet again and again believing that he will make profit in the long run.

By miscalculating probabilities, one can make wrong decisions (and lose a lot of money)!
Solution:

(a) $P[R = -150] = P[N = 1]$ and $P[R = +100] = P[N \neq 1] = P[N = 0] + P[N = 2]$. So,

$$\boxed{p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise.} \end{cases}}$$

Using Alembert's *miscalculated* pmf,

$$p_R(r) = \begin{cases} 1/3, & r = -150, \\ 2/3, & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

(b) From $p_R(r)$ in part (a), we have $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{3} \times (-150) + \frac{2}{3} \times 100 = \boxed{\frac{50}{3}} \approx 16.67$

(c) Again,

$$p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

Using the actual pmf,

$$p_R(r) = \begin{cases} \frac{1}{2}, & r = -150, \\ \frac{1}{4} + \frac{1}{4}, & r = +100, \\ 0, & \text{otherwise} \end{cases} = \boxed{\begin{cases} \frac{1}{2}, & r = -150 \text{ or } +100, \\ 0, & \text{otherwise.} \end{cases}}$$

(d) From $p_R(r)$ in part (c), we have $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{2} \times (-150) + \frac{1}{2} \times 100 = \boxed{-25}$.

(e) Observe that $Y = -R$. Hence, using the answer from part (c), we have

$$p_Y(y) = \boxed{\begin{cases} \frac{1}{2}, & y = +150 \text{ or } -100, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) Observe that $Y = -R$. Hence, $\mathbb{E}Y = -\mathbb{E}R$. Using the actual probabilities, $\mathbb{E}R = -25$ from part (d). Hence, $\mathbb{E}Y = \boxed{+25}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. A random variables X has support containing only two numbers. Its expected value is $\mathbb{E}X = 5$. Its variance is $\operatorname{Var} X = 3$. Give an example of the pmf of such a random variable.

Solution: We first find $\sigma_X = \sqrt{\operatorname{Var} X} = \sqrt{3}$. Recall that this is the average deviation from the mean. Because X takes only two values, we can make them at exactly $\pm\sqrt{3}$ from the mean; that is

$$x_1 = 5 - \sqrt{3} \quad \text{and} \quad x_2 = 5 + \sqrt{3}.$$

In which case, we automatically have $\mathbb{E}X = 5$ and $\operatorname{Var} X = 3$. Hence, one example of such pmf is

$$\boxed{p_X(x) = \begin{cases} \frac{1}{2}, & x = 5 \pm \sqrt{3} \\ 0, & \text{otherwise} \end{cases}}$$

We can also try to find a general formula for x_1 and x_2. If we let $p = P[X = x_2]$, then $q = 1 - p = P[X = x_1]$. Given p, the values of x_1 and x_2 must satisfy two conditions: $\mathbb{E}X = m$ and $\operatorname{Var} X = \sigma^2$. (In our case, $m = 5$ and $\sigma^2 = 3$.) From $\mathbb{E}X = m$, we must have

$$x_1 q + x_2 p = m; \tag{9.2}$$

that is
$$x_1 = \frac{m}{q} - x_2\frac{p}{q}.$$
From $\operatorname{Var} X = \sigma^2$, we have $\mathbb{E}\left[X^2\right] = \operatorname{Var} X + \mathbb{E}X^2 = \sigma^2 + m^2$ and hence we must have
$$x_1^2 q + x_2^2 p = \sigma^2 + m^2. \tag{9.3}$$
Substituting x_1 from (9.2) into (9.3), we have
$$x_2^2 p - 2x_2 mp + \left(pm^2 - q\sigma^2\right) = 0$$
whose solutions are
$$x_2 = \frac{2mp \pm \sqrt{4m^2p^2 - 4p\left(pm^2 - q\sigma^2\right)}}{2p} = \frac{2mp \pm 2\sigma\sqrt{pq}}{2p} = m \pm \sigma\sqrt{\frac{q}{p}}.$$
Using (9.2), we have
$$x_1 = \frac{m}{q} - \left(m \pm \sigma\sqrt{\frac{q}{p}}\right)\frac{p}{q} = m \mp \sigma\sqrt{\frac{p}{q}}.$$
Therefore, for any given p, there are two pmfs:
$$p_X(x) = \begin{cases} 1-p, & x = m - \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m + \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise,} \end{cases}$$
or
$$p_X(x) = \begin{cases} 1-p, & x = m + \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m - \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 6. For each of the following families of random variable X, find the value(s) of x which maximize $p_X(x)$. (This can be interpreted as the "mode" of X.)

(a) $\mathcal{P}(\alpha)$

(b) Binomial(n, p)

(c) $\mathcal{G}_0(\beta)$

(d) $\mathcal{G}_1(\beta)$

Remark [Y&G, p. 66]:

- For statisticians, the mode is the most common number in the collection of observations. There are as many or more numbers with that value than any other value. If there are two or more numbers with this property, the collection of observations is called multimodal. In probability theory, a **mode** of random variable X is a number x_{mode} satisfying
$$p_X(x_{\text{mode}}) \geq p_X(x) \quad \text{for all } x.$$
- For statisticians, the median is a number in the middle of the set of numbers, in the sense that an equal number of members of the set are below the median and above the median. In probability theory, a median, X_{median}, of random variable X is a number that satisfies
$$P[X < X_{\text{median}}] = P[X > X_{\text{median}}].$$
- Neither the mode nor the median of a random variable X need be unique. A random variable can have several modes or medians.

Solution: We first note that when $\alpha > 0$, $p \in (0,1)$, $n \in \mathbb{N}$, and $\beta \in (0,1)$, the above pmf's will be strictly positive for some values of x. Hence, we can discard those x at which $p_X(x) = 0$. The remaining points are all integers. To compare them, we will evaluate $\frac{p_X(i+1)}{p_X(i)}$.

(a) For Poisson pmf, we have
$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{e^{-\alpha}\alpha^{i+1}}{(i+1)!}}{\frac{e^{-\alpha}\alpha^{i}}{i!}} = \frac{\alpha}{i+1}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$ if and only if $i < \alpha - 1$.
- $\frac{p_X(i+1)}{p_X(i)} = 1$ if and only if $i = \alpha - 1$.
- $\frac{p_X(i+1)}{p_X(i)} < 1$ if and only if $i > \alpha - 1$.

Let $\tau = \alpha - 1$. This implies that τ is the place where things change. Moving from i to $i+1$, the probability strictly increases if $i < \tau$. When $i > \tau$, the next probability value (at $i+1$) will decrease.

(i) Suppose $\alpha \in (0,1)$, then $\alpha - 1 < 0$ and hence $i > \alpha - 1$ for all i. (Note that i are are nonnegative integers.) This implies that the pmf is a strictly decreasing function and hence the maximum occurs at the first i which is $i = 0$.

(ii) Suppose $\alpha \in \mathbb{N}$. Then, the pmf will be strictly increasing until we reaches $i = \alpha - 1$. At which point, the next probability value is the same. Then, as we further increase i, the pmf is strictly decreasing. Therefore, the maximum occurs at $\alpha - 1$ and α.

(iii) Suppose $\alpha \notin \mathbb{N}$ and $\alpha \geq 1$. Then we will have have any $i = \alpha - 1$. The pmf will be strictly increasing where the last increase is from $i = \lfloor \alpha - 1 \rfloor$ to $i + 1 = \lfloor \alpha - 1 \rfloor + 1 = \lfloor \alpha \rfloor$. After this, the pmf is strictly decreasing. Hence, the maximum occurs at $\lfloor \alpha \rfloor$.

To summarize,

$$\arg\max_x p_X(x) = \boxed{\begin{cases} 0, & \alpha \in (0,1), \\ \alpha - 1 \text{ and } \alpha, & \alpha \text{ is an integer}, \\ \lfloor \alpha \rfloor, & \alpha > 1 \text{ is not an integer}. \end{cases}}$$

(b) For binomial pmf, we have

$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{n!}{(i+1)!(n-i-1)!} p^{i+1} (1-p)^{n-i-1}}{\frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}} = \frac{(n-i)\,p}{(i+1)\,(1-p)}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$ if and only if $i < np - 1 + p = (n+1)p - 1$.
- $\frac{p_X(i+1)}{p_X(i)} = 1$ if and only if $i = (n+1)p - 1$.
- $\frac{p_X(i+1)}{p_X(i)} < 1$ if and only if $i > (n+1)p - 1$.

Let $\tau = (n+1)p - 1$. This implies that τ is the place where things change. Moving from i to $i+1$, the probability strictly increases if $i < \tau$. When $i > \tau$, the next probability value (at $i+1$) will decrease.

(i) Suppose $(n+1)p$ is an integer. The pmf will strictly increase as a function of i, and then stays at the same value at $i = \tau = (n+1)p - 1$ and $i + 1 = (n+1)p - 1 + 1 = (n+1)p$. Then, it will strictly decrease. So, the maximum occurs at $(n+1)p - 1$ and $(n+1)p$.

(ii) Suppose $(n+1)p$ is not an integer. Then, there will not be any i that is $= \tau$. Therefore, we only have the pmf strictly increases where the last increase occurs when we goes from $i = \lfloor \tau \rfloor$ to $i + 1 = \lfloor \tau \rfloor + 1$. After this, the probability is strictly decreasing. Hence, the maximum is unique and occur at $\lfloor \tau \rfloor + 1 = \lfloor (n+1)p - 1 \rfloor + 1 = \lfloor (n+1)p \rfloor$.

To summarize,

$$\arg\max_x p_X(x) = \boxed{\begin{cases} (n+1)p - 1 \text{ and } (n+1)p, & (n+1)p \text{ is an integer}, \\ \lfloor (n+1)p \rfloor, & (n+1)p \text{ is not an integer}. \end{cases}}$$

(c) $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$. Hence, $p_X(i)$ is strictly decreasing. The maximum occurs at the smallest value of i which is $\boxed{0}$.

(d) $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$. Hence, $p_X(i)$ is strictly decreasing. The maximum occurs at the smallest value of i which is $\boxed{1}$.

Problem 7. An article in Information Security Technical Report ["Malicious Software—Past, Present and Future" (2004, Vol. 9, pp. 618)] provided the data (shown in Figure 9.1) on the top ten malicious software instances for 2002. The clear leader in the number of registered incidences for the year 2002 was the Internet worm "Klez". This virus was first detected on 26 October 2001, and it has held the top spot among malicious software for the longest period in the history of virology.

Place	Name	% Instances
1	I-Worm.Klez	61.22%
2	I-Worm.Lentin	20.52%
3	I-Worm.Tanatos	2.09%
4	I-Worm.BadtransII	1.31%
5	Macro.Word97.Thus	1.19%
6	I-Worm.Hybris	0.60%
7	I-Worm.Bridex	0.32%
8	I-Worm.Magistr	0.30%
9	Win95.CIH	0.27%
10	I-Worm.Sircam	0.24%

Figure 9.1: The 10 most widespread malicious programs for 2002 (Source—Kaspersky Labs).

Suppose that 20 malicious software instances are reported. Assume that the malicious sources can be assumed to be independent.

(a) What is the probability that at least one instance is "Klez"?

(b) What is the probability that three or more instances are "Klez"?

(c) What are the expected value and standard deviation of the number of "Klez" instances among the 20 reported?

Solution: Let N be the number of instances (among the 20) that are "Klez". Then, $N \sim$binomial(n, p) where $n = 20$ and $p = 0.6122$.

(a) $P[N \geq 1] = 1 - P[N < 1] = 1 - P[N = 0] = 1 - p_N(0) = 1 - \binom{20}{0} \times 0.6122^0 \times 0.3878^{20} \approx 0.9999999941 \approx 1$.

(b)
$$\begin{aligned} P[N \geq 3] &= 1 - P[N < 3] = 1 - (P[N = 0] + P[N = 1] + P[N = 2]) \\ &= 1 - \sum_{k=0}^{2} \binom{20}{k} (0.6122)^k (0.3878)^{20-k} \approx 0.999997 \end{aligned}$$

(c) $\mathbb{E}N = np = 20 \times 0.6122 = 12.244$.
$\sigma_N = \sqrt{\operatorname{Var} N} = \sqrt{np(1-p)} = \sqrt{20 \times 0.6122 \times 0.3878} \approx 2.179$.