**ECS 315: Probability and Random Processes** **2018/1**

# HW Solution 7 — Due: Oct 25, 4 PM

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**Problem 1.** For each description of a random variable $X$ below, indicate whether $X$ is a **discrete** random variable.

(a) $X$ is the number of websites visited by a randomly chosen software engineer in a day.

(b) $X$ is the number of classes a randomly chosen student is taking.

(c) $X$ is the average height of the passengers on a randomly chosen bus.

(d) A game involves a circular spinner with eight sections labeled with numbers. $X$ is the amount of time the spinner spins before coming to a rest.

(e) $X$ is the thickness of the longest book in a randomly chosen library.

(f) $X$ is the number of keys on a randomly chosen keyboard.

(g) $X$ is the length of a randomly chosen person's arm.

***Solution***:We consider the number of possibilities for the values of $X$ in each part. If the collection of possible values is countable (finite or countably infinite), then we conclude that the random variable is discrete. Otherwise, the random variable is not discrete. Therefore, the $X$ defined in parts (a), (b), and (f) are discrete. The $X$ defined in other parts are not discrete.

**Problem 2** (Quiz4, 2014)**.** Consider a random experiment in which you roll a 20-sided fair dice. We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega, \quad Y(\omega) = (\omega - 5)^2, \quad Z(\omega) = |\omega - 5| - 3$$

Evaluate the following probabilities:

(a) $P[X = 5]$

(b) $P[Y = 16]$

(c) $P[Y > 10]$

(d) $P[Z > 10]$

(e) $P[5 < Z < 10]$

***Solution***: In this question, $\Omega = \{1, 2, 3, \dots, 20\}$ because the dice has 20 sides. All twenty outcomes are equally-likely because the dice is fair. So, the probability of each outcome is $\frac{1}{20}$:

$$P(\{\omega\}) = \frac{1}{20} \text{ for any } \omega \in \Omega.$$

(a) From $X(\omega) = \omega$, we have $X(\omega) = 5$ if and only if $\omega = 5$.
Therefore, $P[X = 5] = P(\{5\}) = \boxed{\frac{1}{20}}$.

(b) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) = 16$ if and only if $\omega = \pm 4 + 5 = 1$ or $9$.
Therefore, $P[Y = 16] = P(\{1, 9\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(c) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) > 10$ if and only if $(\omega - 5)^2 > 10$.
To check this, it may be more straight-forward to calculate $Y(\omega)$ at all possible values of $\omega$:

| $w$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Y(\omega)$ | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 | 169 | 196 | 225 |

From the table, the values of $\omega$ that satisfy the condition $Y(\omega) > 10$ are $1, 9, 10, 11, \dots, 20$.
Therefore, $P[Y > 10] = P(\{1, 9, 10, 11, \dots, 20\}) = \boxed{\frac{13}{20}}$.

(d) The values of $\omega$ that satisfy $|\omega - 5| - 3 > 10$ are 19 and 20.
To see this, it is straight-forward to calculate $Z(\omega)$ at all possible values of $\omega$:

| $w$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Z(\omega)$ | 1 | 0 | -1 | -2 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Therefore, $P[Z > 10] = P(\{19, 20\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(e) The values of $\omega$ that satisfy $5 < |\omega - 5| - 3 < 10$ are $14, 15, 16, 17$.
Therefore, $P[5 < Z < 10] = P(\{14, 15, 16, 17\}) = \frac{4}{20} = \boxed{\frac{1}{5}}$.

**Problem 3.** Consider the sample space $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$. Suppose that $P(A) = |A|/|\Omega|$ for any event $A \subset \Omega$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of $X$.

***Solution***: The random variable maps the outcomes $\omega = -2, -1, 0, 1, 2, 3, 4$ to numbers $x = 4, 1, 0, 1, 4, 9, 16$, respectively. Therefore,

$$p_X(0) = P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7},$$
$$p_X(1) = P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7},$$
$$p_X(4) = P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7},$$
$$p_X(9) = P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and}$$
$$p_X(16) = P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}.$$

Combining the results above, we get the complete pmf:

$$p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}}$$

**Problem 4.** Suppose $X$ is a random variable whose pmf at $x = 0, 1, 2, 3, 4$ is given by $p_X(x) = \frac{2x+1}{25}$.

Remark: Note that the statement above does not specify the value of the $p_X(x)$ at the value of $x$ that is not 0,1,2,3, or 4.

(a) What is $p_X(5)$?

(b) Determine the following probabilities:

(i) $P[X = 4]$

(ii) $P[X \le 1]$

(iii) $P[2 \le X < 4]$

(iv) $P[X > -10]$

***Solution***:

(a) First, we calculate

$$\sum_{x=0}^{4} p_X(x) = \sum_{x=0}^{4} \frac{2x+1}{25} = \frac{1+3+5+7+9}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other $x$ with $p_X(x) > 0$. At $x = 5$, we then conclude that $p_X(5) = \boxed{0.}$ The same reasoning also implies that $p_X(x) = 0$ at any $x$ that is not 0,1,2,3, or 4.

(b) Recall that, for discrete random variable $X$, the probability

$$P\left[\text{some condition(s) on } X\right]$$

can be calculated by adding $p_X(x)$ for all $x$ in the support of $X$ that satisfies the given condition(s).

(i) $P[X = 4] = p_X(4) = \frac{2\times 4+1}{25} = \boxed{\frac{9}{25}}$.

(ii) $P[X \le 1] = p_X(0) + p_X(1) = \frac{2\times 0+1}{25} + \frac{2\times 1+1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}}$.

(iii) $P[2 \le X < 4] = p_X(2) + p_X(3) = \frac{2\times 2+1}{25} + \frac{2\times 3+1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}}$.

(iv) $P[X > -10] = \boxed{1}$ because all the $x$ in the support of $X$ satisfies $x > -10$.

**Problem 5.** The random variable $V$ has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant $c$.

(b) Find $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}]$.

(c) Find the probability that $V$ is an even number.

(d) Find $P[V > 2]$.

(e) Sketch $p_V(v)$.

(f) Sketch $F_V(v)$. (Note that $F_V(v) = P[V \le v]$.)

***Solution***: [Y&G, Q2.2.3]

(a) We choose $c$ so that the pmf sums to one:

$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

Hence, $c = \boxed{1/30}$.

(b) $P[V \in \{u^2 : u = 1, 2, 3, \ldots\}] = P[V \in \{1, 4, 9, 16, 25\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$.

(c) $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$.

(d) $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$.

(e) See Figure 7.1 for the sketch of $p_V(v)$:

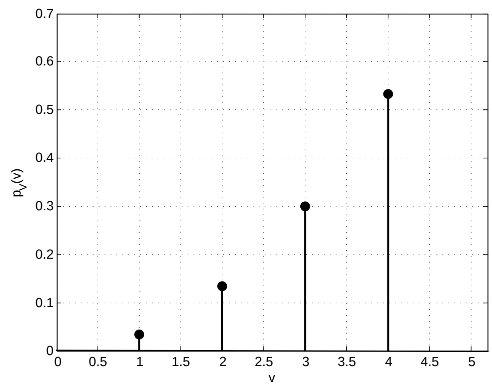


Figure 7.1: Sketch of $p_V(v)$ for Question 5

(f) See Figure 7.2 for the sketch of $F_V(v)$:

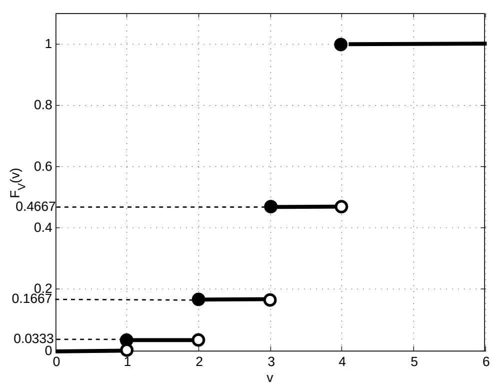


Figure 7.2: Sketch of $F_V(v)$ for Question 5

**Problem 6.** The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8}, \\ 0.2, & \frac{1}{8} \le x < \frac{1}{4}, \\ 0.9, & \frac{1}{4} \le x < \frac{3}{8}, \\ 1 & x \ge \frac{3}{8}. \end{cases}$$

Determine the following probabilities:

(a) $P[X \le 1/18]$

(b) $P[X \le 1/4]$

(c) $P[X \le 5/16]$

(d) $P[X > 1/4]$

(e) $P[X \le 1/2]$

[Montgomery and Runger, 2010, Q3-42]

Remark: Try to calculate these values directly from the cdf. (Avoid converting the cdf to pmf first.)

***Solution***:

(a) $P[X \le 1/18] = F_X(1/18) = \boxed{0}$ because $\frac{1}{18} < \frac{1}{8}$.

(b) $P[X \le 1/4] = F_X(1/4) = \boxed{0.9}$.

(c) $P[X \le 5/16] = F_X(5/16) = \boxed{0.9}$ because $\frac{1}{4} < \frac{5}{16} < \frac{3}{8}$.

(d) $P[X > 1/4] = 1 - P[X \le 1/4] = 1 - F_X(1/4) = 1 - 0.9 = \boxed{0.1}$.

(e) $P[X \le 1/2] = F_X(1/2) = \boxed{1}$ because $\frac{1}{2} > \frac{3}{8}$.

Alternatively, we can also derive the pmf first and then calculate the probabilities.