ECS 315: Probability and Random Processes **2018/1**

HW Solution 4 — Due: September 18, 4 PM

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Problem 1. Continue from Problem 2 in HW3.

Recall that, there, we consider a random experiment whose sample space is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Find the following probabilities.

(a) $P(A|B)$

(b) $P(B|A)$

(c) $P(B|A^c)$

Solution: In HW3, we have already found

$$\begin{aligned} P(A) &= P(\{a, b, c\}) = 0.1 + 0.1 + 0.2 = 0.4, \\ P(B) &= P(\{c, d, e\}) = 0.2 + 0.4 + 0.2 = 0.8, \text{ and} \\ P(A \cap B) &= P(\{c\}) = 0.2. \end{aligned}$$

Therefore, by definition,

(a) $P(A|B) \equiv \frac{P(A\cap B)}{P(B)} = \frac{0.2}{0.8} = \boxed{\frac{1}{4}}$ and

(b) $P(B|A) \equiv \frac{P(B\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)} = \frac{0.2}{0.4} = \boxed{\frac{1}{2}}$.

(c) DO ***NOT*** start with $P(B|A^c) = 1 - P(B|A)$. This is not one of the formulas for conditional probabilities. Here, we will have to go back to the definition:

$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)} = \frac{P(\{d, e\})}{P(\{d, e\})} = \boxed{1}.$$

Problem 2.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a)

(i) By definition, $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.4 \times 0.5 = \boxed{0.2.}$$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, one can apply the property $P(A^c|B) = 1 - P(A|B)$ to get

$$P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3.$$

(b) **Method 1**: By definition, $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.2 \times 0.8 = 0.16.$$

Next, from $P(B) = 0.8$, we know that

$$P(B^c) = 1 - P(B) = 1 - 0.8 = 0.2.$$

By definition, $P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)}$. Therefore,

$$P(A \cap B^c) = P(A|B^c)P(B^c) = 0.3 \times 0.2 = 0.06.$$

Hence, $P(A) = P(A \cap B) + P(A \cap B^c) = 0.16 + 0.16 = \boxed{0.22.}$

Method 2: By the total probability formula, $P(A) = P(A|B)P(B)+P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$.

Method 3: For those who are not seeking a "smart" way to solve this question, we can try the following:

Note that when we have two events, the sample space is always partitioned into four events: $A \cap B$, $A^c \cap B$, $A \cap B^c$, and $A^c \cap B^c$. (It might be helpful to draw the Venn diagram here.) Let's define their probabilities as p_1, p_2, p_3, and p_4, respectively. We are given three conditions which can then be turned into three equations. There is also one extra condition that $p_1 + p_2 + p_3 + p_4 = 1$. Therefore, we have four equations with four unknowns. Applying some high-school algebra, we should be able to solve for p_1, p_2, p_3, and p_4. With these, we can calculate probability of any event. For example, $P(A) = p_1 + p_3$.

Problem 3. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

(a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

(a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

(b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c) = 0.99 \times 0.0002 + 0.01 \times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+) = \frac{0.99 \times 0.0002}{0.99 \times 0.0002 + 0.01 \times 0.9998} \approx \boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 4. Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped. [Gubner, 2006, Ex.1.20]

(a) Find the probability that a packet is dropped.
Hint: Use total probability theorem.

(b) Find the conditional probability that a packet is routed through El Paso given that it is not dropped.
Hint: Use Bayes' theorem.

Solution: To solve this problem, we use the notation $E = \{\text{routed through El Paso}\}$ and $D = \{\text{packet is dropped}\}$. With this notation, it is easy to interpret the problem as telling us that

$$P(D|E) = 1/3, \quad P(D|E^c) = 1/4, \quad \text{and } P(E) = 3/4.$$

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|E^c)P(E^c) = (1/3)(3/4) + (1/4)(1-3/4) \\ &= 1/4 + 1/16 = \boxed{5/16} = 0.3125. \end{aligned}$$

(b) $P(E|D^c) = \frac{P(E\cap D^c)}{P(D^c)} = \frac{P(D^c|E)P(E)}{P(D^c)} = \frac{(1-1/3)(3/4)}{1-5/16} = \boxed{\frac{8}{11}} \approx 0.7273.$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. Someone has rolled a fair dice twice. Suppose he tells you that "one of the rolls turned up a face value of six". What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Note the followings:

- The answer is not $\frac{1}{6}$.
- Although there is no use of the word "given" or "conditioned on" in this question, the probability we seek is a conditional one. We have an extra piece of information because we know that the event "one of the rolls turned up a face value of six" has occurred.
- The question says "one of the rolls" without telling us which roll (the first or the second) it is referring to.

Solution: Let the sample space be the set $\{(i,j)|i,j = 1,\dots,6\}$, where i and j denote the outcomes of the first and second rolls, respectively. They are all equally likely; so each has probability of $1/36$. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6),\dots,(5,6),(6,6),(6,5),\dots,(6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A\cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Problem 6.

(a) Suppose that $P(A|B) = 1/3$ and $P(A|B^c) = 1/4$. Find the range of the possible values for $P(A)$.

(b) Suppose that C_1, C_2, and C_3 partition Ω. Furthermore, suppose we know that $P(A|C_1) = 1/3$, $P(A|C_2) = 1/4$ and $P(A|C_3) = 1/5$. Find the range of the possible values for $P(A)$.

Solution: First recall the total probability theorem: Suppose we have a collection of events $B_1, B_2, \ldots, B_n$ which partitions Ω. Then,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots P(A|B_n)P(B_n) \end{aligned}$$

(a) Note that B and B^c partition Ω. So, we can apply the total probability theorem:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)).$$

You may check that, by varying the value of $P(B)$ from 0 to 1, we can get the value of $P(A)$ to be any number in the range $\left[\frac{1}{4}, \frac{1}{3}\right]$. Technically, we can not use $P(B) = 0$ because that would make $P(A|B)$ not well-defined. Similarly, we can not use $P(B) = 1$ because that would mean $P(B^c) = 0$ and hence make $P(A|B^c)$ not well-defined. Therfore, the range of $P(A)$ is $\boxed{\left(\frac{1}{4}, \frac{1}{3}\right).}$

Note that larger value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) < \frac{1}{3}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{3}.$$

Similarly, smaller value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) > \frac{1}{4}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{4}.$$

(b) Again, we apply the total probability theorem:

$$\begin{aligned} P(A) &= P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + P(A|C_3)P(C_3) \\ &= \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3). \end{aligned}$$

Because C_1, C_2, and C_3 partition Ω, we know that $P(C_1) + P(C_2) + P(C_3) = 1$. Now,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) < \frac{1}{3}P(C_1) + \frac{1}{3}P(C_2) + \frac{1}{3}P(C_3) = \frac{1}{3}.$$

Similarly,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) > \frac{1}{5}P(C_1) + \frac{1}{5}P(C_2) + \frac{1}{5}P(C_3) = \frac{1}{5}.$$

Therefore, $P(A)$ must be inside $\left(\frac{1}{5}, \frac{1}{3}\right)$.

You may check that any value of $P(A)$ in the range $\boxed{\left(\frac{1}{5}, \frac{1}{3}\right)}$ can be obtained by first setting the value of $P(C_2)$ to be close to 0 and varying the value of $P(C_1)$ from 0 to 1.