ECS 315: Probability and Random Processes **2018/1**

HW Solution 3 — Due: September 11, 4 PM

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Problem 1. If A, B, and C are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P((A \cup B) \cap C)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

Solution:

(a) Because A, B, and C are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because A, B, and C are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because A and B are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among A, B, and C, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$

Problem 2. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

Solution:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a,b,c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned} P(B) &= P(\{c,d,e\}) = P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.} \end{aligned}$$

(c) Applying the complement rule, we have $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

Problem 3. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

(c) Use the binomial theorem (3.2) to evaluate $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k}$.

Solution:

(a) The coefficient of $x^r y^{n-r}$ is $\binom{n}{r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} = \boxed{5,200,300}$.

(b) We start from the expansion of $(a+b)^n$. Then we set $a = 2x$ and $b = -3y$:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (2x)^r (-3y)^{n-r} = . \sum_{r=0}^{n} \binom{n}{r} 2^r (-3)^{n-r} x^r y^{n-r}. \tag{3.2}$$

Therefore, the coefficient of $x^r y^{n-r}$ is $\binom{n}{r} 2^r (-3)^{n-r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} 2^{12} (-3)^{13} = -\frac{25!}{12!13!} 2^{12} 3^{13} = \boxed{-33959763545702400}$.

(c) From (3.2), set $x = -1$ and $y = 1$, then we have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (-1+1)^n = \boxed{0}$.

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[1] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

[1] This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.3}$$

(a) Use the binomial theorem (3.2) to simplify the following sums

(i) $\sum\limits_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(ii) $\sum\limits_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(b) If we differentiate (3.2) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r \binom{n}{r} x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r}$.

Solution:

(a) To deal with the sum involving only the even terms (or only the odd terms), we first use (3.2) to expand $(x+y)^n$ and $(x+(-y))^n$. When we add the expanded results, only the even terms in the sum are left. Similarly, when we find the difference between the two expanded results, only the the odd terms are left. More specifically,

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n + (y-x)^n\right), \text{ and}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n - (y-x)^n\right).$$

If $x + y = 1$, then

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 + (1-2x)^n\right)}, \text{ and} \tag{3.4a}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 - (1-2x)^n\right)}. \tag{3.4b}$$

(b) $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r} = \boxed{nx\left(x(n-1)(x+y)^{n-2} + (x+y)^{n-1}\right)}$.

Problem 6.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A \cap B)$.

First, recall[2], from the lecture notes, that "$P(A \cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.5}$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{3.6}$$

Now, $P(A \cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \le P(A \cup B) \le 1 \tag{3.7}$$

Combining (**??**) with (**??**) gives

$$P(A) + P(B) - 1 \le P(A \cap B) \le P(A) + P(B). \tag{3.8}$$

The second inequality in (**??**) is not useful because (**??**) gives a better[3] bound. So, we will replace the second inequality with (**??**):

$$P(A) + P(B) - 1 \le P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.9}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \le P(A \cap B) \le 1 \tag{3.10}$$

Combining (**??**) and (**??**), we have

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B), 1\}.$$

[2] Again, to see this, note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$.

[3] When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A)+P(B)-1),0\} \leq P(A\cap B) \leq \min\{P(A),P(B)\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6},\frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A\subset B$. The lower-bound can be obtained by considering an example where $A\cup B=\Omega$.

(b) We will try to derive general bounds for $P(A\cup B)$.

By monotonicity, because both A and B are subset of $A\cup B$, we must have

$$P(A\cup B) \geq \max\{P(A),P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A\cup B) \leq P(A)+P(B).$$

Therefore,

$$\max\{P(A),P(B)\} \leq P(A\cup B) \leq P(A)+P(B).$$

Being a probability, $P(A\cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A),P(B),0\} \leq P(A\cup B) \leq \min\left\{\left(P(A)+P(B)\right),1\right\}.$$

Note that number 0 is not needed in the minimization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A),P(B)\} \leq P(A\cup B) \leq \min\{(P(A)+P(B)),1\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{3}$, we have

$$P(A\cup B) \in \boxed{\left[\frac{1}{2},\frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A\perp B$. The lower-bound is achieved when $B\subset A$.

Problem 7. (**Classical Probability and Combinatorics**) Suppose n integers are chosen with replacement (that is, the same integer could be chosen repeatedly) at random from $\{1, 2, 3, \ldots, N\}$. Calculate the probability that the chosen numbers arise according to some non-decreasing sequence.

Solution: There are N^n possible sequences. (This is ordered sampling with replacement.) To find the probability, we need to count the number of non-decreasing sequences among these N^n possible sequences. It takes some thought to realize that this is exactly the counting problem that we called "unordered sampling with replacement". In which case, we can conclude that the probability is $\boxed{\frac{\binom{n+N-1}{n}}{N^n}}$. The "with replacement" part should be clear from the question statement. The "unordered" part needs some more thought.

To see this, let's look back at how we turn the "ordered sampling *without replacement*" into "unordered sampling *without replacement*". Recall that there are $(N)_n$ distinct samples for "ordered sampling without replacement". When we switch to the "unordered" case, we see that many of the original samples from the "ordered sampling without replacement" are regarded as the same in the "unordered" case. In fact, we can form "groups" of samples whose members are regarded as the same in the "unordered" case. We can then count the number of groups. In class, we found that the size of any individual group can be calculated easily from permuting the elements in a sample and hence there are $n!$ members in each group. This leads us to conclude that there are $(N)_n/n! = \binom{N}{n}$ groups.

We are in a similar situation when we want to turn the "ordered sampling *with replacement*" into "unordered sampling *with replacement*". We first start with N^n distinct samples from "ordered sampling with replacement". Now, we again separate these samples into groups. Let's consider an example where $n = 3$. Then sequences "1 1 2", "1 2 1", and "2 1 1" are put together in the same group in the "unordered" case. Note that the size of this group is 3. The sequences "1 2 3", "1 3 2", "2 1 3", "2 3 1", "3 1 2", and "3 2 1" are in another group. Note that the size of this group is 6. Therefore, the group sizes are not the same and hence we can not find the number of groups by $N^n/(\text{group size})$ as in the sampling *without replacement* discussed in the previous paragraph. To count the number of groups, we look at the sequences from another perspective. We see that the "unordered" case, the only information the characterizes each group is "how many of each number there are". This is why we can match the number of groups to the number of nonnegative-integer solutions to the equation $x_1 + x_2 + \cdots + x_N = n$ as discussed in class. Finally, note that for each group, we have only one possible nondecreasing sequence. So, the number of possible nondecreasing sequence is the same as the number of groups.

If you think about the explanation above, you may realize that, by requiring the "order" on the sequence, the counting problem become "unordered sampling".

Here, we present two direct methods that leads to the same answer.

Method 1: Because the sequence is non-decreasing, the number of times that each of the integers $\{1, 2, \ldots, N\}$ shows up in the sequence is the only information that characterizes each sequence. Let x_i be the number of times that number i shows up in the sequence. The number of sequences is then the same as the number of solution to the equation $x_1+x_2+\cdots+x_N = n$ where the x_i are all non-negative integers. We have seen in class that the number of solutions is $\binom{n+N-1}{n}$.

Method 2: [DasGupta, 2010, Example 1.14, p. 12] Consider the following construction of such non-decreasing sequence. Start with n stars and $N-1$ bars. There are $\binom{n+N-1}{n}$ arrangements of these. For example, when $N = 5$ and $n = 2$, one arrangement is $| * || * |$. Now, add spaces between these bars and stars including before the first one and after the last one. For our earlier example, we have $\_|\_ * \_|\_|\_ * \_|\_$. Now, put number 1 in the leftmost space. After this position, the next space holds the same value as the previous on if you pass a $*$. On the other hand, if you pass a $|$ then the value increases by 1. Note that because there are $N-1$ bars, the last space always gets the value N. What you now have is a sequence of $n+N$ numbers with bars between consecutive distinct numbers and stars between consecutive equal numbers. For example, our example would gives $1|2*2|3|4*4|5$. Note that this gives a non-decreasing sequence of $n+N$ numbers. The corresponding non-decreasing sequence of n numbers for this arrangement of stars and bars is (2,4); that is we only take the numbers to the right of the stars. Because there are n stars, our sequence will have n numbers. It will be non-decreasing because it is a sub-sequence of the non-decreasing $n+N$ sequence. This shows that any arrangement of n stars and $N-1$ bars gives one nondecreasing sequence of n numbers.

Conversely, we can take any nondecreasing sequence of n numbers and combine it with the full set of numbers $\{1, 2, 3, \ldots, N\}$ to form a set of $n+N$ numbers. Now rearrange these numbers in a nondecreasing order. Put a bar between consecutive distinct numbers in this set and a star between consecutive equal numbers in this set. Note that the number to the right of each star is an element of the original n-number sequence. This shows that any nondecreasing sequence of n numbers corresponds to an arrangement of n stars and $N-1$ bars.

Combining the two paragraphs above, we now know that the number of non-decreasing sequences is the same as the number of arrangement of n stars and $N-1$ bars, which is $\binom{n+N-1}{n}$.

Remark: There is also a method— which will not be discussed here, but can be inferred by finding the pattern of the sums that lead to the number of non-decreasing sequences as we increase the value of n— that would interestingly give the number of non-decreasing sequences as

$$\sum_{k_{n-1}=1}^{N} \cdots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} k_1.$$

This sum can be simplified into $\binom{n+N-1}{n}$ by the "parallel summation formula" which is well-known but we didn't discuss in class because this is not a class on combinatorics.