Name ________________________ ID3 _ _ _

ECS 315: Probability and Random Processes **2018/1**

HW 12 — Due: Not Due

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Problem 1. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{c|c} x \backslash y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0.02 & 0.10 & 0.08 \\ 0.08 & 0.32 & 0.40 \end{bmatrix} \end{array}$$

(a) Evaluate the following quantities:

(i) The marginal pmf $p_X(x)$

(ii) The marginal pmf $p_Y(y)$

(iii) $\mathbb{E}X$

(iv) $\operatorname{Var} X$

(v) $\mathbb{E}Y$

(vi) $\operatorname{Var} Y$

(vii) $P[XY < 6]$

(viii) $P[X = Y]$

(ix) $\mathbb{E}[XY]$

(x) $\mathbb{E}[(X-3)(Y-2)]$

(xi) $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$

(xii) $\text{Cov}[X, Y]$

(xiii) $\rho_{X,Y}$

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\text{Var}\, X$, $\text{Cov}[X, Y]$, and $\rho_{X,Y}$ that you got earlier.

(i) $\text{Cov}[3X + 4, 6Y - 7]$

(ii) $\rho_{3X+4,6Y-7}$

(iii) $\text{Cov}[X, 6X - 7]$

(iv) $\rho_{X,6X-7}$

Problem 2. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

(a) $\mathbb{E}[(X - 3)(Y - 2)]$

(b) $\text{Cov}[X, Y]$

(c) $\rho_{X,Y}$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 3. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

(c) Find the expected value of X.

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

Problem 4. Let X be a uniform random variable on the interval $[0,1]$. Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

Are the events $[X \in A], [X \in B]$, and $[X \in C]$ independent?

Problem 5. Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120–240 mg/dl. The Food and Nutrition Institute of the Philippines found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200 mg/dl. Suppose that the cholesterol level in the population is normally distributed.

(a) Determine the standard deviation of this distribution.

(b) What is the value of the cholesterol level that exceeds 90% of the population?

(c) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion?

(d) An adult is thought to be at high risk if his cholesterol level is more than two standard deviations above the mean. What percentage of the population is at high risk?

Problem 6 (Q3.5.6)**.** Solve this question using the Φ/Q table.

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of n years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable Y_n with expected value $40n$ and variance $100n$.

(a) What is the probability that Y_{20} exceeds 1000?

(b) How many years n must the professor teach in order that $P[Y_n > 1000] > 0.99$?

Problem 7. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Problem 8. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

```
x = 0:5; pX = binopdf(x,5,1/3).
```

What is the expression that would give `pY`, a corresponding vector describing the pmf of Y?

(b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use "for-loop", "while-loop", "if statement". Hint: Multiply them in an appropriate orientation.

(c) Use `MATLAB` to evaluate the following quantities. Again, do not use "for-loop", "while-loop", "if statement".

(i) $\mathbb{E}X$

(ii) $P[X = Y]$

(iii) $P[XY < 6]$

Problem 9. Suppose $\operatorname{Var} X = 5$. Find $\operatorname{Cov}[X, X]$ and $\rho_{X,X}$.

Problem 10. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

(a) Find $\text{Var}[X+Y]$.

(b) Find $\mathbb{E}\left[(Y-3X+5)^2\right]$. Assume $\mathbb{E}\left[Y-3X+5\right] = 1$.

Problem 11. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x,y)$, where $x = 1,2,3$ and $y = 1,2,3,4,5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i,j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\text{Var}\, X$

(e) Find $\text{Var}\, Y$