ECS 315: Probability and Random Processes **2018/1**

HW Solution 11 — Due: Not Due

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Problem 1 (Yates and Goodman, 2005, Q3.4.5). X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

(e) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & x > b. \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $F_X(x) = \boxed{\begin{cases} 0, & x < -5, \\ \frac{x+5}{10}, & a \le x \le b. \\ 1, & x > 5 \end{cases}}$

(c) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\, dx = \int\limits_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^2}{2}\right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}$.

In general,

$$\mathbb{E}X = \int\limits_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(d) $\mathbb{E}[X^5] = \int\limits_{-\infty}^{\infty} x^5 f_X(x)\,dx = \int\limits_{-5}^{5} x^5 \times \frac{1}{10} dx = \frac{1}{10}\left.\frac{x^6}{6}\right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}$.

In general,

$$\mathbb{E}\left[X^5\right] = \int\limits_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a}\int\limits_a^b x^5 dx = \frac{1}{b-a}\left.\frac{x^6}{6}\right|_a^b = \frac{1}{b-a}\frac{b^6-a^6}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}[X^5] = \boxed{0}$.

(e) In general,

$$\mathbb{E}\left[e^X\right] = \int\limits_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a}\int\limits_a^b e^x dx = \frac{1}{b-a}\left. e^x\right|_a^b = \frac{e^b - e^a}{b-a}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$.

Problem 2 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t+\theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

(b) Y is another function of Θ.

$$\mathbb{E}[Y] = \mathbb{E}[5\cos(7t_1+\Theta)\times 5\cos(7t_2+\Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1+\theta)\times 5\cos(7t_2+\theta)d\theta$$
$$= \frac{25}{2\pi}\int_0^{2\pi} \cos(7t_1+\theta)\times\cos(7t_2+\theta)d\theta.$$

Recall[1] the cosine identity

$$\cos(a)\times\cos(b) = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$

Therefore,

$$\mathbb{E}Y = \frac{25}{4\pi}\int_0^{2\pi} \cos(7t_1+7t_2+2\theta)+\cos(7(t_1-t_2))\,d\theta$$
$$= \frac{25}{4\pi}\left(\int_0^{2\pi}\cos(7t_1+7t_2+2\theta)\,d\theta + \int_0^{2\pi}\cos(7(t_1-t_2))\,d\theta\right).$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1-t_2))\int_0^{2\pi}d\theta = \frac{25}{4\pi}\cos(7(t_1-t_2))\,2\pi = \boxed{\frac{25}{2}\cos(7(t_1-t_2))}.$$

Problem 3. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constant m and positive number σ. Furthermore, when a Gaussian random variable has $m = 0$ and $\sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1-\Phi(\cdot)$) are traditionally provided by a table.

Suppose Z is a standard Gaussian random variable.

[1]This identity could be derived easily via the Euler's identity:

$$\cos(a)\times\cos(b) = \frac{e^{ja}+e^{-ja}}{2}\times\frac{e^{jb}+e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb}+e^{-ja}e^{jb}+e^{ja}e^{-jb}+e^{-ja}e^{-jb}\right)$$
$$= \frac{1}{2}\left(\frac{e^{ja}e^{jb}+e^{-ja}e^{-jb}}{2}+\frac{e^{-ja}e^{jb}+e^{ja}e^{-jb}}{2}\right)$$
$$= \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$

(a) Use the Φ table to find the following probabilities:

(i) $P[Z < 1.52]$
(ii) $P[Z < -1.52]$
(iii) $P[Z > 1.52]$
(iv) $P[Z > -1.52]$
(v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

(i) $P[Z > c] = 0.14$
(ii) $P[-c < Z < c] = 0.95$

Solution:

(a)

(i) $P[Z < 1.52] = \Phi(1.52) = \boxed{0.9357.}$
(ii) $P[Z < -1.52] = \Phi(-1.52) = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$
(iii) $P[Z > 1.52] = 1 - P[Z < 1.52] = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$
(iv) It is straightforward to see that the area of $P[Z > -1.52]$ is the same as $P[Z < 1.52] = \Phi(1.52)$. So, $P[Z > -1.52] = \boxed{0.9357.}$
Alternatively, $P[Z > -1.52] = 1 - P[Z \leq -1.52] = 1 - \Phi(-1.52) = 1 - (1 - \Phi(1.52)) = \Phi(1.52)$.
(v) $P[-1.36 < Z < 1.52] = \Phi(1.52) - \Phi(-1.36) = \Phi(1.52) - (1 - \Phi(1.36)) = \Phi(1.52) + \Phi(1.36) - 1 = 0.9357 + 0.9131 - 1 = \boxed{0.8488.}$

(b)

(i) $P[Z > c] = 1 - P[Z \leq c] = 1 - \Phi(c)$. So, we need $1 - \Phi(c) = 0.14$ or $\Phi(c) = 1 - 0.14 = 0.86$. In the Φ table, we do not have exactly 0.86, but we have 0.8599 and 0.8621. Because 0.86 is closer to 0.8599, we answer the value of c whose $\phi(c) = 0.8599$. Therefore, $c \approx \boxed{1.08.}$
(ii) $P[-c < Z < c] = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1$. So, we need $2\Phi(c) - 1 = 0.95$ or $\Phi(c) = 0.975$. From the Φ table, we have $c \approx \boxed{1.96.}$

Problem 4. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function.

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(e) Observe that the Φ table ("Table 4" from the lecture) stops at $z = 2.99$ and the Q table ("Table 5" from the lecture) starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Solution:

(a) Recall that when $X \sim \mathcal{N}(m, \sigma^2)$, $F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$. Here, $T \sim \mathcal{N}(85, 10^2)$. Therefore, $F_T(t) = \boxed{\Phi\left(\frac{t-85}{10}\right).}$

(b)

(i) $P[T > 100] = 1 - P[T \le 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) = 1 - \Phi(1.5)$

(ii) $P[T < 60] = P[T \le 60]$ because T is a continuous random variable and hence $P[T = 60] = 0$. Now, $P[T \le 60] = F_T(60) = \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) = \boxed{1 - \Phi(2.5).}$ Note that, for the last equality, we use the fact that $\Phi(-z) = 1 - \Phi(z)$.

(iii)

$$\begin{aligned} P[70 \le T \le 100] = F_T(100) - F_T(70) &= \Phi\left(\frac{100-85}{10}\right) - \Phi\left(\frac{70-85}{10}\right) \\ &= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - (1 - \Phi(1.5)) = \boxed{2\Phi(1.5) - 1.} \end{aligned}$$

(c) In this question, we use the fact that $Q(x) = 1 - \Phi(x)$.

(i) $1 - \Phi(1.5) = \boxed{Q(1.5).}$

(ii) $1 - \Phi(2.5) = \boxed{Q(2.5).}$

(iii) $2\Phi(1.5) - 1 = 2(1 - Q(1.5)) - 1 = 2 - 2Q(1.5) - 1 = \boxed{1 - 2Q(1.5).}$

(d)

(i) $1 - \Phi(1.5) = 1 - 0.9332 = \boxed{0.0668.}$

(ii) $1 - \Phi(2.5) = 1 - 0.99379 = \boxed{0.0062.}$

(iii) $2\Phi(1.5) - 1 = 2(0.9332) - 1 = \boxed{0.8664.}$

(e) When z is large, $\Phi(z)$ will start with 0.999... The first few significant digits will all be the same and hence not quite useful to be there.

Problem 5. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Solution: Let T be the time to failure (in hours). We are given that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3 \times 10^{-4}$. Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P[T > 10^4]$.

We shall first provide the general formula for the ccdf $P[T > t]$ when $t > 0$:

$$P[T > t] = \int_t^\infty f_T(\tau)d\tau = \int_t^\infty \lambda e^{-\lambda\tau} d\tau = -e^{-\lambda\tau}\Big|_t^\infty = e^{-\lambda t}. \tag{11.1}$$

Therefore,

$$P\left[T > 10^4\right] = e^{-3\times10^{-4}\times10^4} = \boxed{e^{-3} \approx 0.0498.}$$

(b) We start with $P[T \le 7000] = 1 - P[T > 7000]$. Next, we apply (**??**) to get

$$P[T \le 7000] = 1 - P[T > 7000] = 1 - e^{-3\times10^{-4}\times7000} = \boxed{1 - e^{-2.1} \approx 0.8775.}$$