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ECS 315: Probability and Random Processes **2018/1**

HW 10 — Due: Nov 22, 4 PM

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Instructions

(a) This assignment has 6 pages.

(b) (1 pt) Work and write your answers **directly on these sheets** (not on other blank sheets of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all problems.

(e) Carefully write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1 (Yates and Goodman, 2005, Q3.2.1)**.** The random variable X has probability density function

unknown constant

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Use the pdf to find the following quantities.

(a) the constant c Recall that any pdf should integrate to 1.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx\,dx = c\int_0^2 x\,dx = c\left.\frac{x^2}{2}\right|_0^2 = \underbrace{2c}_{\text{This should } = 1.}$$

Therefore, $c = \frac{1}{2}$.

(b) $P[0 \le X \le 1]$

$$= \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \frac{1}{4}.$$

(c) $P[-1/2 \le X \le 1/2]$. $= \int_{-1/2}^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{1}{2}x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^{1/2} = \frac{1}{16}$.

$f_X(x) = 0$ on $[-\frac{1}{2}, 0)$

(d) the cdf $F_X(x)$.

For $x<0$, because $f_X(t)=0$ for $t<0$, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0$

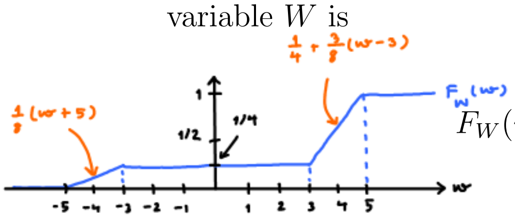
For $0 \le x \le 2$, $f_X(t) = \frac{t}{2}$ and $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{t}{2}\,dt = \frac{t^2}{4}\Big|_0^x = \frac{x^2}{4}$.

At $x=2$, $F_X(2) = 1$.

For $x>2$, $f_X(t)=0$. Therefore, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \underbrace{\int_{-\infty}^{2} f_X(t)\,dt}_{F_X(2)=1} + \underbrace{\int_{2}^{x} \underbrace{f_X(t)}_{0}\,dt}_{0} = 1$.

$$F_X(x) = \begin{cases} 0, & x<0, \\ x^2/4, & 0 \le x \le 2, \\ 1, & \text{otherwise.} \end{cases}$$

Problem 2 (Modified from Yates and Goodman, 2005, Q3.1.3)**.** The CDF of a random variable W is



$$F_W(w) = \begin{cases} 0, & w < -5, \\ (w+5)/8, & -5 \le w < -3, \\ 1/4, & -3 \le w < 3, \\ 1/4 + 3(w-3)/8, & 3 \le w < 5, \\ 1, & w \ge 5. \end{cases}$$

Remark: It is possible to solve this problem by finding the pdf first. (You are asked to derive the pdf anyway in the enxt problem.) However, you should also make sure that you know how to calculate the probabilities directly from the cdf.

(a) Is W a continuous random variable?

From the plot above, we see that $F_W(w)$ is a continuous function.

Because its cdf is continuous, we conclude that W is a continuous RV.

(b) What is $P[W \le 4]$?

$P[W \le 4] = F_W(4) = \frac{1}{4} + \frac{3}{8}(4-3) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \approx 0.625$

by definition of cdf

(c) What is $P[-2 < W \le 2]$?

$P[-2 < W \le 2] = F_W(2) - F_W(-2) = \frac{1}{4} - \frac{1}{4} = 0$

For continuous RV, $P[a \le X \le b] = F_X(b) - F_X(a)$

(d) What is $P[W > 0]$?

$P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(A) = 1 - P(A^c)$

(e) What is the value of a such that $P[W \le a] = 1/2$?

$P[W \le a] = F_W(a)$. From the plot above, we know that to have $F_W(a) = \frac{1}{2}$, the value of a must be in the interval $(3,5)$.

In this interval, $F_W(a) = \frac{1}{4} + \frac{3}{8}(a-3)$.

So, we solve for "a" that satisfies $\frac{1}{4} + \frac{3}{8}(a-3) = \frac{1}{2} \Rightarrow a = \frac{11}{3} \approx 3.67$

Problem 3 (Yates and Goodman, 2005, Q3.2.3). The CDF of random variable W is

$$F_W(w) = \begin{cases} 0, & w < -5, \\ (w+5)/8, & -5 \le w < -3, \\ 1/4, & -3 \le w < 3, \\ 1/4 + 3(w-3)/8, & 3 \le w < 5, \\ 1, & w \ge 5. \end{cases}$$

Find its pdf $f_W(w)$.

Given a cdf, we can find the pdf by taking derivative.

As discussed in class, for the location(s) where derivative does not exist, we can choose to define the pdf to be any convenient value.

In this question, the cdf is given in the form of expressions on several intervals. It is then easy to find its derivative inside each of the intervals:

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 0, & w < -5, \\ 1/8, & -5 < w < -3, \\ 0, & -3 < w < 3, \\ 3/8, & 3 < w < 5, \\ 0, & 5 < w. \end{cases}$$

It should be clear from the plot of cdf in the previous problem that the derivative does not exist at $w = -5, -3, 3, 5$. We choose to assign $f_W(w) = 0$ at these points.

$$f_W(w) = \begin{cases} 1/8, & -5 < w < -3 \\ 3/8, & 3 < w < 5 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4 (Yates and Goodman, 2005, Q3.3.4)**.** The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Solution:

(a) Recall that, for continuous random variable Y,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when y is outside of the interval $[0,2)$, $f_Y(y) = 0$ and hence does not affect the integration. We only need to integrate over $[0,2)$ in which $f_Y(y) = \frac{y}{2}$. Therefore,

$$\mathbb{E}Y = \int_0^2 y\left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^2}{2} dy = \left.\frac{y^3}{2\times 3}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(b) The variance of any random variable Y (discrete or continuous) can be found from

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2$$

We have already calculate $\mathbb{E}Y$ in the previous part. So, now we need to calculate $\mathbb{E}[Y^2]$. Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy.$$

Here, $g(y) = y^2$. Therefore,

$$\mathbb{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy.$$

Again, in the integration, we can ignore the y whose $f_Y(y) = 0$:

$$\mathbb{E}\left[Y^2\right] = \int_0^2 y^2\left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^3}{2} dy = \left.\frac{y^4}{2\times 4}\right|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

Problem 5 (Yates and Goodman, 2005, Q3.3.6)**.** The cdf of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144, & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $f_V(v)$?

(b) What is $\mathbb{E}[V]$?

(c) What is $\mathrm{Var}[V]$?

(d) What is $\mathbb{E}[V^3]$?

Solution: First, let's check whether V is a continuous random variable. This can be done easily by checking whether its cdf $F_V(v)$ is a continuous function. The cdf of V is defined using three expressions. Note that each expression is a continuous function. So, we only need to check whether there is/are any jump(s) at the boundaries: $v = 5$ and $v = 7$. Plugging $v = 5$ into $(v+5)^2/144$ gives 0 which matches the value of the expression for $v < -5$. Plugging $v = 7$ into $(v+5)^2/144$ gives 1 which matches the value of the expression for $v \ge 7$. SO, there is no discontinuity in $F_V(v)$. It is a continuous function and hence V itself is a continuous random variable.

(a) We can find the pdf $f_V(v)$ at almost all of the v by finding the derivative of the cdf $F_V(v)$:

$$f_V(v) = \frac{d}{dv} F_V(v) = \begin{cases} 0, & v < -5, \\ \frac{v+5}{72}, & -5 < v < 7, \\ 0, & v > 7. \end{cases}$$

Note that we still haven't specified $f_V(v)$ at $v = 5$ and $v = 7$. This is because the formula for $F_V(v)$ changes at those points and hence to actually find the derivatives, we would need to look at both the left and right derivatives at these points. The derivative may not even exist there. The good news is that we don't have to actually find them because $v = 5$ and $v = 7$ correspond to just two points on the pdf. Because V is a continuous random variable, we can "define" or "set " $f_V(v)$ to be any values there. In this case, for brevity of the expression, let's set the pdf to be 0 there. This gives

$$f_V(v) = \frac{d}{dv} F_V(v) = \boxed{\begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) $\mathbb{E}[V] = \int\limits_{-\infty}^{\infty} v f_V(v)\, dv = \int\limits_{-5}^{7} v \left(\frac{v+5}{72}\right) dv = \frac{1}{72} \int\limits_{-5}^{7} v^2 + 5v dv = \boxed{3}.$

(c) $\mathbb{E}\left[V^2\right] = \int\limits_{-\infty}^{\infty} v^2 f_V\left(v\right) dv = \int\limits_{-5}^{7} v^2 \left(\frac{v+5}{72}\right) dv = 17.$

Therefore, $\operatorname{Var} V = \mathbb{E}\left[V^2\right] - \left(\mathbb{E}\left[V\right]\right)^2 = 17 - 9 = \boxed{8}$.

(d) $\mathbb{E}\left[V^3\right] = \int\limits_{-\infty}^{\infty} v^3 f_V\left(v\right) dv = \int\limits_{-5}^{7} v^3 \left(\frac{v+5}{72}\right) dv = \boxed{\frac{431}{5} = 86.2}$.