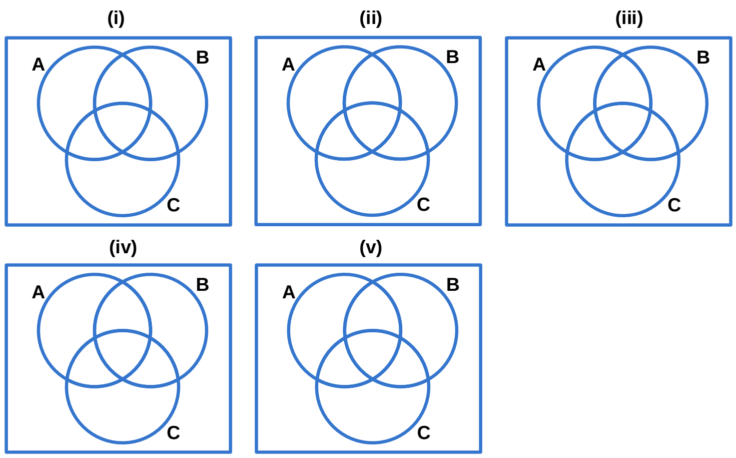
ECS 315: Probability and Random Processes **2018/1**

HW Solution 1 — Due: August 28, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. (**Set Theory**) For this problem, only answers are needed; you don't have to describe your solution.

(a) In the Venn diagrams below,



shade the region that corresponds to the following events:

(i) A^c

(ii) $A \cap B$

(iii) $(A \cap B) \cup C$

(iv) $(B \cup C)^c$

(v) $(A \cap B)^c \cup C$

[Montgomery and Runger, 2010, Q2-19]

(b) Let $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and put $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6\}$. Find

(i) $A \cup B$

(ii) $A \cap B$

(iii) $A \cap C$

(iv) A^c

(v) $B \setminus A$

Solution:

(a) See Figure 1.1

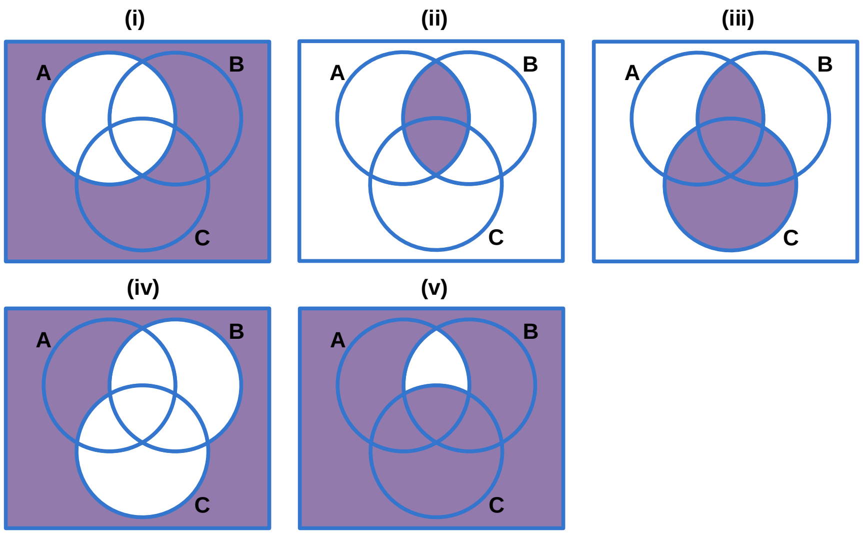


Figure 1.1: Venn diagrams for events in Problem 1

(b) $A \cup B = \{1, 2, 3, 4, 5, 6\}$, $A \cap B = \{3, 4\}$, $A \cap C = \emptyset$, $B \setminus A = \{5, 6\} = C$.

Problem 2. For this problem, only answers are needed; you don't have to provide explanation.

For each of the sets provided in the first column of the table below, indicate (by putting a Y(es) or an N(o) in the appropriate cells of the table) whether it is "finite", "infinite", "countable", "countably infinite", "uncountable".

Sets	Finite	Infinite	Countable	Countably Infinite	Uncountable
$\{1\}$					
$\{1, 2\}$					
$[1, 2]$					
$[1, 2] \cup [-1, 0]$					
$\{1, 2, 3, 4\}$					
the power set of $\{1, 2, 3, 4\}$					
the set of all real numbers					
the set of all real-valued x satisfying $\cos x = 0$					
the set of all integers					
$(-\infty, 0]$					
$(-\infty, 0] \cap [0, +\infty)$					

Solution: First, note that the intersection in the last row can be simplified into a singleton $\{0\}$. Being an intersection of intervals may make it look like an uncountable sets. However, only one number survives the intersection.

The sets $\{1\}$, $\{1, 2\}$, $\{1, 2, 3, 4\}$, $2^{\{1,2,3,4\}}$, and $\{0\}$ are all finite set because their size (cardinality) are finite. Because they are finite, they are not infinite. Any finite set is countable. So, they are countable. They are not infinite; so they can't be countably infinite nor uncountable. Their corresponding rows should be Y N Y N N.

The x that satisfies $\cos x = 0$ is any x of the form $\frac{\pi}{2} + k\pi$ where k is any integer. So, the collection of these x has the same size (cardinality) as the set of all integers. They are countably infinite which means they are infinite and countable. Because they are countable, they are not uncountable. Because they are infinite, they are not finite. Their corresponding rows should be N Y Y Y N.

The sets $[1, 2]$, $[1, 2] \cup [-1, 0]$, $\mathbb{R}$, and $(-\infty, 0]$ are all intervals and hence uncountable. Uncountable sets are not countable and hence can't be countably infinite. They are infinite and hence not finite. Their corresponding rows should be N Y N N Y.

Sets	Finite	Infinite	Countable	Countably Infinite	Uncountable
$\{1\}$	Y	N	Y	N	N
$\{1,2\}$	Y	N	Y	N	N
$[1,2]$	N	Y	N	N	Y
$[1,2]\cup[-1,0]$	N	Y	N	N	Y
$\{1,2,3,4\}$	Y	N	Y	N	N
the power set of $\{1,2,3,4\}$	Y	N	Y	N	N
the set of all real numbers	N	Y	N	N	Y
the set of all real-valued x satisfying $\cos x = 0$	N	Y	Y	Y	N
the set of all integers	N	Y	Y	Y	N
$(-\infty,0]$	N	Y	N	N	Y
$(-\infty,0]\cap[0,+\infty)$	Y	N	Y	N	N

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HW Solution 2 — Due: September 4, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. [Montgomery and Runger, 2010, Q2-54] Each of the possible five outcomes of a random experiment is equally likely. The sample space is $\{a, b, c, d, e\}$. Let A denote the event $\{a, b\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

Solution: Because the outcomes are equally likely, we can simply use classical probability.

(a) $P(A) = \frac{|A|}{|\Omega|} = \boxed{\frac{2}{5}}$

(b) $P(B) = \frac{|B|}{|\Omega|} = \boxed{\frac{3}{5}}$

(c) $P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{5-2}{5} = \boxed{\frac{3}{5}}$

(d) $P(A \cup B) = \frac{|\{a,b,c,d,e\}|}{|\Omega|} = \frac{5}{5} = \boxed{1}$

(e) $P(A \cap B) = \frac{|\emptyset|}{|\Omega|} = \boxed{0}$

Problem 2. (**Classical Probability and Combinatorics**) Shuffle a deck of cards and cut it into three piles. What is the probability that (at least) a court card will turn up on top of one of the piles.

Hint: There are 12 court cards (four jacks, four queens and four kings) in the deck.

Solution: In [Lovell, 2006, p. 17–19], this problem is named "Three Lucky Piles".

Method 1: When somebody cuts three piles, they are, in effect, randomly picking three cards from the deck. There are $52 \times 51 \times 50$ possible outcomes. The number of outcomes that do not contain any court card is $40 \times 39 \times 38$. So, the probability of having at least one court card is

$$\boxed{\frac{52 \times 51 \times 50 - 40 \times 39 \times 38}{52 \times 51 \times 50} \approx 0.553.}$$

Method 2: Note that our solution above, especially the part where we use the words "in effect", may not be so evident to some of you. If you want to solve this question directly, you can approach it using the total probability theorem which is studied in Chapter 6. In the beginning, we shuffle the cards. So, after the shuffling, we will have a deck of 52 cards with all the possible 52! permutations being equally likely. (In our mind,) we label the cards with #1 to #52 from the top to bottom. Now, the next step is to cut it into three piles. Note that this is the same as choosing two cards (from #2 (top) to #52 (bottom)) to indicate where the two boundaries (which are the same as the two cards at the top of second and third piles) are. Note also that this process is usually biased. Most will try to divide the deck into three piles of approximately equal size. So, it is *unlikely* that you will have the first piles with 50 cards, the second with only one card, and the third with only one card. So, classical probability can not be used here. We only know that there are $\binom{51}{2} = 1,275$ ways to perform the cutting for a particular deck and they are not equally likely. Let event $B_1, \ldots, B_{1275}$ denote each of these cases. For example, B_{134} may be the case in which the cutting positions are at cards #32 and #45. So, the top cards on the three piles are cards #1, #32, and #45. Let A be the event that at least one of these cards is a court card. Of course, the "at least one" counting problem can be simplified by considering the opposite case. A^c is the event that none of the three top cards is a court card. So, there are $52 - 12 = 40$ choices for card #1. There are $40-1 = 39$ choices for card #32. There are $39-1 = 38$ choices for card #45. For the remaining $52 - 3 = 49$ cards, there is no restriction. So, there are 49! choices. In total, we have $40 \times 39 \times 38 \times (49!)$ shuffled patterns among the 52! equally likely possibilities that satisfy A^c. Therefore,

$$P\left(A \left| B_{134} \right.\right) = \frac{52! - 40 \times 39 \times 38 \times (49!)}{52!} = 1 - \frac{40 \times 39 \times 38}{52 \times 51 \times 50} \approx 0.553.$$

The same reasoning applies to any cutting positions. So, $P\left(A \left| B_k \right.\right) \approx 0.553$ for any k. By the total probability theorem,

$$P(A) = \sum_{k=1}^{1275} P\left(A \left| B_k \right.\right) P(B_k) \approx \sum_{k=1}^{1275} 0.553 P(B_k) = 0.553 \sum_{k=1}^{1275} P(B_k) = 0.553 \times 1 = 0.553.$$

Observe that we still don't know the value of each $P(B_k)$ but we know that the sum of them is 1.

Problem 3. (**Classical Probability**) There are three buttons which are painted red on one side and white on the other. If we tosses the buttons into the air, calculate the probability that all three come up the same color.

Remarks: A *wrong* way of thinking about this problem is to say that there are four ways they can fall. All red showing, all white showing, two reds and a white or two whites and a red. Hence, it seems that out of four possibilities, there are two favorable cases and hence the probability is $1/2$.

Solution: There are 8 possible outcomes. (The same number of outcomes as tossing three coins.) Among these, only two outcomes will have all three buttons come up the same color. So, the probability is $2/8 = \boxed{1/4}$.

Problem 4. (**Classical Probability and Combinatorics**) A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases.

(a) How many different designs are possible? [Montgomery and Runger, 2010, Q2-51]

(b) A specific design is randomly generated by the Web server when you visit the site. If you visit the site five times, what is the probability that you will not see the same design? [Montgomery and Runger, 2010, Q2-71]

Solution:

(a) By the multiplication rule, total number of possible designs

$$= 4 \times 3 \times 5 \times 3 \times 5 = \boxed{900}.$$

(b) From part (a), total number of possible designs is 900. The sample space is now the set of all possible designs that may be seen on five visits. It contains $(900)^5$ outcomes. (This is ordered sampling with replacement.)

The number of outcomes in which all five visits are different can be obtained by realizing that this is ordered sampling without replacement and hence there are $(900)_5$ outcomes. (Alternatively, On the first visit any one of 900 designs may be seen. On the second visit there are 899 remaining designs. On the third visit there are 898 remaining designs. On the fourth and fifth visits there are 897 and 896 remaining designs, respectively. From the multiplication rule, the number of outcomes where all designs are different is $900 \times 899 \times 898 \times 897 \times 896$.)

Therefore, the probability that a design is not seen again is

$$\frac{(900)_5}{900^5} \approx \boxed{0.9889.}$$

Problem 5. (**Classical Probability and Combinatorics**) A bin of 50 parts contains five that are defective. A sample of two parts is selected at random, without replacement. Determine the probability that both parts in the sample are defective. [Montgomery and Runger, 2010, Q2-49]

Solution: The number of ways to select two parts from 50 is $\binom{50}{2}$ and the number of ways to select two defective parts from the 5 defective ones is $\binom{5}{2}$ Therefore the probability is

$$\frac{\binom{5}{2}}{\binom{50}{2}} = \frac{2}{245} = \boxed{0.0082}.$$

Alternatively, if the two parts in the sample are selected one by one, then we may also consider their ordering as well. In such case, we use the formula for "ordered sampling without replacement" instead of "unordered sampling without replacement":

$$\frac{(5)_2}{(50)_2} = \frac{5 \times 4}{50 \times 49} = \frac{2}{245} = \boxed{0.0082}.$$

Problem 6. (**Combinatorics**) Consider the design of a communication system in the United States.

(a) How many three-digit phone prefixes that are used to represent a particular geographic area (such as an area code) can be created from the digits 0 through 9?

(b) How many three-digit phone prefixes are possible in which no digit appears more than once in each prefix?

(c) As in part (a), how many three-digit phone prefixes are possible that do not start with 0 or 1, but contain 0 or 1 as the middle digit?

[Montgomery and Runger, 2010, Q2-45]

Solution:

(a) From the multiplication rule (or by realizing that this is ordered sampling with replacement), $10^3 = \boxed{1,000}$ prefixes are possible

(b) This is ordered sampling without replacement. Therefore $(10)_3 = 10 \times 9 \times 8 = \boxed{720}$ prefixes are possible

(c) From the multiplication rule, $8 \times 2 \times 10 = \boxed{160}$ prefixes are possible.

Problem 7. (**Classical Probability and Combinatorics**) We all know that the chance of a head (H) or tail (T) coming down after a fair coin is tossed are fifty-fifty. If a fair coin is tossed ten times, then intuition says that five heads are likely to turn up.

Calculate the probability of getting exactly five heads (and hence exactly five tails).

Solution: There are 2^{10} possible outcomes for ten coin tosses. (For each toss, there is two possibilities, H or T). Only $\binom{10}{5}$ among these outcomes have exactly heads and five tails. (Choose 5 positions from 10 position for H. Then, the rest of the positions are automatically T.) The probability of have exactly 5 H and 5 T is

$$\boxed{\frac{\binom{10}{5}}{2^{10}} \approx 0.246.}$$

Note that five heads and five tails will turn up more frequently than any other single combination (one head, nine tails for example) but the sum of all the other possibilities is much greater than the single 5 H, 5 T combination.

Extra Question

Here is an optional question for those who want more practice.

Problem 8. ***An Even Split at Coin Tossing***: Let p_n be the probability of getting exactly n heads (and hence exactly n tails) when a fair coin is tossed $2n$ times.

(a) Find p_n.

(b) Sometimes, to work theoretically with large factorials, we use Stirling's Formula:

$$n! \approx \sqrt{2\pi n}n^n e^{-n} = \left(\sqrt{2\pi e}\right) e^{\left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)}. \tag{2.1}$$

Approximate p_n using Stirling's Formula.

(c) Find $\lim_{n\to\infty} p_n$.

Solution: Note that we have worked on a particular case ($n = 5$) of this problem earlier.

(a) Use the same solution as Problem 7; change 5 to n and 10 to $2n$, we have

$$p_n = \boxed{\frac{\binom{2n}{n}}{2^{2n}}}.$$

(b) By Stirling's Formula, we have

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \approx \frac{\sqrt{2\pi 2n}(2n)^{2n}e^{-2n}}{\left(\sqrt{2\pi n}n^n e^{-n}\right)^2} = \frac{4^n}{\sqrt{\pi n}}.$$

Hence,

$$p_n \approx \boxed{\frac{1}{\sqrt{\pi n}}}. \tag{2.2}$$

[Mosteller, *Fifty Challenging Problems in Probability with Solutions*, 1987, Problem 18]

See Figure 2.1 for comparison of p_n and its approximation via Stirling's formula.

(c) From (2.2), $\lim_{n\to\infty} p_n = \boxed{0}$. A more rigorous proof of this limit would use the bounds

$$\frac{4^n}{\sqrt{4n}} \le \binom{2n}{n} \le \frac{4^n}{\sqrt{3n+1}}.$$

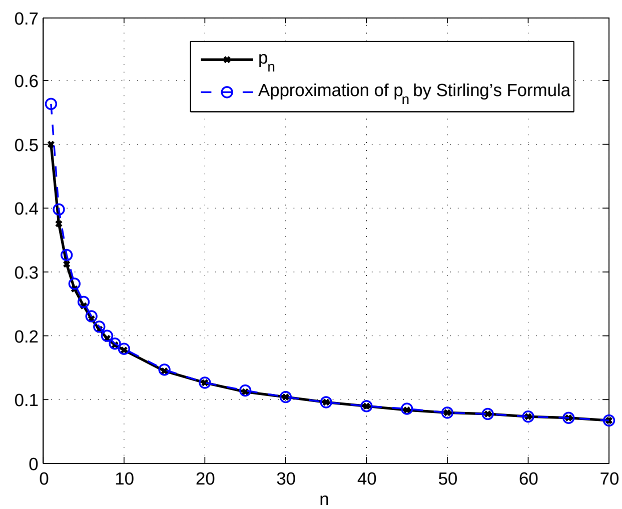

Figure 2.1: Comparison of p_n and its approximation via Stirling's formula

ECS 315: Probability and Random Processes **2018/1**

HW Solution 3 — Due: September 11, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. If A, B, and C are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P((A \cup B) \cap C)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

Solution:

(a) Because A, B, and C are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because A, B, and C are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because A and B are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among A, B, and C, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$

Problem 2. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

Solution:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a,b,c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned} P(B) &= P(\{c,d,e\}) = P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.} \end{aligned}$$

(c) Applying the complement rule, we have $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

Problem 3. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

(c) Use the binomial theorem (3.2) to evaluate $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

Solution:

(a) The coefficient of $x^r y^{n-r}$ is $\binom{n}{r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} = \boxed{5,200,300}$.

(b) We start from the expansion of $(a+b)^n$. Then we set $a = 2x$ and $b = -3y$:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (2x)^r (-3y)^{n-r} = . \sum_{r=0}^{n} \binom{n}{r} 2^r (-3)^{n-r} x^r y^{n-r}. \tag{3.2}$$

Therefore, the coefficient of $x^r y^{n-r}$ is $\binom{n}{r} 2^r (-3)^{n-r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} 2^{12} (-3)^{13} = -\frac{25!}{12!13!} 2^{12} 3^{13} = \boxed{-33959763545702400}$.

(c) From (3.2), set $x = -1$ and $y = 1$, then we have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (-1+1)^n = \boxed{0}$.

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[1] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

[1]This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.3}$$

(a) Use the binomial theorem (3.2) to simplify the following sums

(i) $\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(ii) $\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(b) If we differentiate (3.2) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r \binom{n}{r} x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r}$.

Solution:

(a) To deal with the sum involving only the even terms (or only the odd terms), we first use (3.2) to expand $(x+y)^n$ and $(x+(-y))^n$. When we add the expanded results, only the even terms in the sum are left. Similarly, when we find the difference between the two expanded results, only the the odd terms are left. More specifically,

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n + (y-x)^n\right), \text{ and}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n - (y-x)^n\right).$$

If $x + y = 1$, then

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 + (1-2x)^n\right)}, \text{ and} \tag{3.4a}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 - (1-2x)^n\right)}. \tag{3.4b}$$

(b) $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r} = \boxed{nx\left(x(n-1)(x+y)^{n-2} + (x+y)^{n-1}\right)}$.

Problem 6.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A \cap B)$.

First, recall[2], from the lecture notes, that "$P(A \cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.5}$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{3.6}$$

Now, $P(A \cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \le P(A \cup B) \le 1 \tag{3.7}$$

Combining (**??**) with (**??**) gives

$$P(A) + P(B) - 1 \le P(A \cap B) \le P(A) + P(B). \tag{3.8}$$

The second inequality in (**??**) is not useful because (**??**) gives a better[3] bound. So, we will replace the second inequality with (**??**):

$$P(A) + P(B) - 1 \le P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.9}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \le P(A \cap B) \le 1 \tag{3.10}$$

Combining (**??**) and (**??**), we have

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B), 1\}.$$

[2]Again, to see this, note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$.

[3]When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A)+P(B)-1),0\} \leq P(A\cap B) \leq \min\{P(A),P(B)\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6},\frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

(b) We will try to derive general bounds for $P(A\cup B)$.

By monotonicity, because both A and B are subset of $A\cup B$, we must have

$$P(A\cup B) \geq \max\{P(A),P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A\cup B) \leq P(A)+P(B).$$

Therefore,

$$\max\{P(A),P(B)\} \leq P(A\cup B) \leq P(A)+P(B).$$

Being a probability, $P(A\cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A),P(B),0\} \leq P(A\cup B) \leq \min\left\{\left(P(A)+P(B)\right),1\right\}.$$

Note that number 0 is not needed in the minimization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A),P(B)\} \leq P(A\cup B) \leq \min\{(P(A)+P(B)),1\}.$$

Plugging in the value $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{3}$, we have

$$P(A\cup B) \in \boxed{\left[\frac{1}{2},\frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.

Problem 7. (**Classical Probability and Combinatorics**) Suppose n integers are chosen with replacement (that is, the same integer could be chosen repeatedly) at random from $\{1, 2, 3, \ldots, N\}$. Calculate the probability that the chosen numbers arise according to some non-decreasing sequence.

Solution: There are N^n possible sequences. (This is ordered sampling with replacement.) To find the probability, we need to count the number of non-decreasing sequences among these N^n possible sequences. It takes some thought to realize that this is exactly the counting problem that we called "unordered sampling with replacement". In which case, we can conclude that the probability is $\boxed{\frac{\binom{n+N-1}{n}}{N^n}}$. The "with replacement" part should be clear from the question statement. The "unordered" part needs some more thought.

To see this, let's look back at how we turn the "ordered sampling *without replacement*" into "unordered sampling *without replacement*". Recall that there are $(N)_n$ distinct samples for "ordered sampling without replacement". When we switch to the "unordered" case, we see that many of the original samples from the "ordered sampling without replacement" are regarded as the same in the "unordered" case. In fact, we can form "groups" of samples whose members are regarded as the same in the "unordered" case. We can then count the number of groups. In class, we found that the size of any individual group can be calculated easily from permuting the elements in a sample and hence there are $n!$ members in each group. This leads us to conclude that there are $(N)_n/n! = \binom{N}{n}$ groups.

We are in a similar situation when we want to turn the "ordered sampling *with replacement*" into "unordered sampling *with replacement*". We first start with N^n distinct samples from "ordered sampling with replacement". Now, we again separate these samples into groups. Let's consider an example where $n = 3$. Then sequences "1 1 2", "1 2 1", and "2 1 1" are put together in the same group in the "unordered" case. Note that the size of this group is 3. The sequences "1 2 3", "1 3 2", "2 1 3", "2 3 1", "3 1 2", and "3 2 1" are in another group. Note that the size of this group is 6. Therefore, the group sizes are not the same and hence we can not find the number of groups by $N^n/(\text{group size})$ as in the sampling *without replacement* discussed in the previous paragraph. To count the number of groups, we look at the sequences from another perspective. We see that the "unordered" case, the only information the characterizes each group is "how many of each number there are". This is why we can match the number of groups to the number of nonnegative-integer solutions to the equation $x_1 + x_2 + \cdots + x_N = n$ as discussed in class. Finally, note that for each group, we have only one possible nondecreasing sequence. So, the number of possible nondecreasing sequence is the same as the number of groups.

If you think about the explanation above, you may realize that, by requiring the "order" on the sequence, the counting problem become "unordered sampling".

Here, we present two direct methods that leads to the same answer.

Method 1: Because the sequence is non-decreasing, the number of times that each of the integers $\{1, 2, \ldots, N\}$ shows up in the sequence is the only information that characterizes each sequence. Let x_i be the number of times that number i shows up in the sequence. The number of sequences is then the same as the number of solution to the equation $x_1+x_2+\cdots+x_N = n$ where the x_i are all non-negative integers. We have seen in class that the number of solutions is $\binom{n+N-1}{n}$.

Method 2: [DasGupta, 2010, Example 1.14, p. 12] Consider the following construction of such non-decreasing sequence. Start with n stars and $N-1$ bars. There are $\binom{n+N-1}{n}$ arrangements of these. For example, when $N = 5$ and $n = 2$, one arrangement is $| * || * |$. Now, add spaces between these bars and stars including before the first one and after the last one. For our earlier example, we have $\_|\_*\_|\_|\_*\_|\_$. Now, put number 1 in the leftmost space. After this position, the next space holds the same value as the previous on if you pass a $*$. On the other hand, if you pass a $|$ then the value increases by 1. Note that because there are $N-1$ bars, the last space always gets the value N. What you now have is a sequence of $n+N$ numbers with bars between consecutive distinct numbers and stars between consecutive equal numbers. For example, our example would gives $1|2*2|3|4*4|5$. Note that this gives a non-decreasing sequence of $n+N$ numbers. The corresponding non-decreasing sequence of n numbers for this arrangement of stars and bars is (2,4); that is we only take the numbers to the right of the stars. Because there are n stars, our sequence will have n numbers. It will be non-decreasing because it is a sub-sequence of the non-decreasing $n+N$ sequence. This shows that any arrangement of n stars and $N-1$ bars gives one nondecreasing sequence of n numbers.

Conversely, we can take any nondecreasing sequence of n numbers and combine it with the full set of numbers $\{1, 2, 3, \ldots, N\}$ to form a set of $n+N$ numbers. Now rearrange these numbers in a nondecreasing order. Put a bar between consecutive distinct numbers in this set and a star between consecutive equal numbers in this set. Note that the number to the right of each star is an element of the original n-number sequence. This shows that any nondecreasing sequence of n numbers corresponds to an arrangement of n stars and $N-1$ bars.

Combining the two paragraphs above, we now know that the number of non-decreasing sequences is the same as the number of arrangement of n stars and $N-1$ bars, which is $\binom{n+N-1}{n}$.

Remark: There is also a method— which will not be discussed here, but can be inferred by finding the pattern of the sums that lead to the number of non-decreasing sequences as we increase the value of n— that would interestingly give the number of non-decreasing sequences as

$$\sum_{k_{n-1}=1}^{N} \cdots \sum_{k_2=1}^{k_3} \sum_{k_1=1}^{k_2} k_1.$$

This sum can be simplified into $\binom{n+N-1}{n}$ by the "parallel summation formula" which is well-known but we didn't discuss in class because this is not a class on combinatorics.

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HW Solution 4 — Due: September 18, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Continue from Problem 2 in HW3.

Recall that, there, we consider a random experiment whose sample space is $\{a,b,c,d,e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a,b,c\}$, and let B denote the event $\{c,d,e\}$. Find the following probabilities.

(a) $P(A|B)$

(b) $P(B|A)$

(c) $P(B|A^c)$

Solution: In HW3, we have already found

$$\begin{aligned} P(A) &= P(\{a,b,c\}) = 0.1+0.1+0.2 = 0.4, \\ P(B) &= P(\{c,d,e\}) = 0.2+0.4+0.2 = 0.8, \text{ and} \\ P(A\cap B) &= P(\{c\}) = 0.2. \end{aligned}$$

Therefore, by definition,

(a) $P(A|B) \equiv \frac{P(A\cap B)}{P(B)} = \frac{0.2}{0.8} = \boxed{\frac{1}{4}}$ and

(b) $P(B|A) \equiv \frac{P(B\cap A)}{P(A)} = \frac{P(A\cap B)}{P(A)} = \frac{0.2}{0.4} = \boxed{\frac{1}{2}}$.

(c) DO ***NOT*** start with $P(B|A^c) = 1 - P(B|A)$. This is not one of the formulas for conditional probabilities. Here, we will have to go back to the definition:

$$P(B|A^c) = \frac{P(B\cap A^c)}{P(A^c)} = \frac{P(\{d,e\})}{P(\{d,e\})} = \boxed{1}.$$

Problem 2.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A\cap B)$

(ii) $P(A^c\cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a)

(i) By definition, $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.4 \times 0.5 = \boxed{0.2.}$$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, one can apply the property $P(A^c|B) = 1 - P(A|B)$ to get

$$P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3.$$

(b) **Method 1**: By definition, $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.2 \times 0.8 = 0.16.$$

Next, from $P(B) = 0.8$, we know that

$$P(B^c) = 1 - P(B) = 1 - 0.8 = 0.2.$$

By definition, $P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)}$. Therefore,

$$P(A \cap B^c) = P(A|B^c)P(B^c) = 0.3 \times 0.2 = 0.06.$$

Hence, $P(A) = P(A \cap B) + P(A \cap B^c) = 0.16 + 0.16 = \boxed{0.22.}$

Method 2: By the total probability formula, $P(A) = P(A|B)P(B)+P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$.

Method 3: For those who are not seeking a "smart" way to solve this question, we can try the following:

Note that when we have two events, the sample space is always partitioned into four events: $A \cap B$, $A^c \cap B$, $A \cap B^c$, and $A^c \cap B^c$. (It might be helpful to draw the Venn diagram here.) Let's define their probabilities as p_1, p_2, p_3, and p_4, respectively. We are given three conditions which can then be turned into three equations. There is also one extra condition that $p_1 + p_2 + p_3 + p_4 = 1$. Therefore, we have four equations with four unknowns. Applying some high-school algebra, we should be able to solve for p_1, p_2, p_3, and p_4. With these, we can calculate probability of any event. For example, $P(A) = p_1 + p_3$.

Problem 3. Suppose that for the general population, 1 in 5000 people carries the human immunodeficiency virus (HIV). A test for the presence of HIV yields either a positive (+) or negative (-) response. Suppose the test gives the correct answer 99% of the time.

(a) What is $P(-|H)$, the conditional probability that a person tests negative given that the person does have the HIV virus?

(b) What is $P(H|+)$, the conditional probability that a randomly chosen person has the HIV virus given that the person tests positive?

Solution:

(a) Because the test is correct 99% of the time,

$$P(-|H) = P(+|H^c) = \boxed{0.01}.$$

(b) Using Bayes' formula, $P(H|+) = \frac{P(+|H)P(H)}{P(+)}$, where $P(+)$ can be evaluated by the total probability formula:

$$P(+) = P(+|H)P(H) + P(+|H^c)P(H^c) = 0.99 \times 0.0002 + 0.01 \times 0.9998.$$

Plugging this back into the Bayes' formula gives

$$P(H|+) = \frac{0.99 \times 0.0002}{0.99 \times 0.0002 + 0.01 \times 0.9998} \approx \boxed{0.0194}.$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 2%. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 4. Due to an Internet configuration error, packets sent from New York to Los Angeles are routed through El Paso, Texas with probability 3/4. Given that a packet is routed through El Paso, suppose it has conditional probability 1/3 of being dropped. Given that a packet is not routed through El Paso, suppose it has conditional probability 1/4 of being dropped. [Gubner, 2006, Ex.1.20]

(a) Find the probability that a packet is dropped.
Hint: Use total probability theorem.

(b) Find the conditional probability that a packet is routed through El Paso given that it is not dropped.
Hint: Use Bayes' theorem.

Solution: To solve this problem, we use the notation $E = \{\text{routed through El Paso}\}$ and $D = \{\text{packet is dropped}\}$. With this notation, it is easy to interpret the problem as telling us that

$$P(D|E) = 1/3, \quad P(D|E^c) = 1/4, \quad \text{and } P(E) = 3/4.$$

(a) By the law of total probability,

$$\begin{aligned} P(D) &= P(D|E)P(E) + P(D|E^c)P(E^c) = (1/3)(3/4) + (1/4)(1-3/4) \\ &= 1/4 + 1/16 = \boxed{5/16} = 0.3125. \end{aligned}$$

(b) $P(E|D^c) = \frac{P(E\cap D^c)}{P(D^c)} = \frac{P(D^c|E)P(E)}{P(D^c)} = \frac{(1-1/3)(3/4)}{1-5/16} = \boxed{\frac{8}{11}} \approx 0.7273.$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. Someone has rolled a fair dice twice. Suppose he tells you that "one of the rolls turned up a face value of six". What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Note the followings:

- The answer is not $\frac{1}{6}$.
- Although there is no use of the word "given" or "conditioned on" in this question, the probability we seek is a conditional one. We have an extra piece of information because we know that the event "one of the rolls turned up a face value of six" has occurred.
- The question says "one of the rolls" without telling us which roll (the first or the second) it is referring to.

Solution: Let the sample space be the set $\{(i,j)|i,j = 1,\dots,6\}$, where i and j denote the outcomes of the first and second rolls, respectively. They are all equally likely; so each has probability of $1/36$. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6),\dots,(5,6),(6,6),(6,5),\dots,(6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A\cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Problem 6.

(a) Suppose that $P(A|B) = 1/3$ and $P(A|B^c) = 1/4$. Find the range of the possible values for $P(A)$.

(b) Suppose that C_1, C_2, and C_3 partition Ω. Furthermore, suppose we know that $P(A|C_1) = 1/3$, $P(A|C_2) = 1/4$ and $P(A|C_3) = 1/5$. Find the range of the possible values for $P(A)$.

Solution: First recall the total probability theorem: Suppose we have a collection of events $B_1, B_2, \ldots, B_n$ which partitions Ω. Then,

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_n) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots P(A|B_n)P(B_n) \end{aligned}$$

(a) Note that B and B^c partition Ω. So, we can apply the total probability theorem:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)).$$

You may check that, by varying the value of $P(B)$ from 0 to 1, we can get the value of $P(A)$ to be any number in the range $\left[\frac{1}{4}, \frac{1}{3}\right]$. Technically, we can not use $P(B) = 0$ because that would make $P(A|B)$ not well-defined. Similarly, we can not use $P(B) = 1$ because that would mean $P(B^c) = 0$ and hence make $P(A|B^c)$ not well-defined. Therfore, the range of $P(A)$ is $\boxed{\left(\frac{1}{4}, \frac{1}{3}\right).}$

Note that larger value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) < \frac{1}{3}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{3}.$$

Similarly, smaller value of $P(A)$ is not possible because

$$P(A) = \frac{1}{3}P(B) + \frac{1}{4}(1 - P(B)) > \frac{1}{4}P(B) + \frac{1}{3}(1 - P(B)) = \frac{1}{4}.$$

(b) Again, we apply the total probability theorem:

$$\begin{aligned} P(A) &= P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + P(A|C_3)P(C_3) \\ &= \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3). \end{aligned}$$

Because C_1, C_2, and C_3 partition Ω, we know that $P(C_1) + P(C_2) + P(C_3) = 1$. Now,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) < \frac{1}{3}P(C_1) + \frac{1}{3}P(C_2) + \frac{1}{3}P(C_3) = \frac{1}{3}.$$

Similarly,

$$P(A) = \frac{1}{3}P(C_1) + \frac{1}{4}P(C_2) + \frac{1}{5}P(C_3) > \frac{1}{5}P(C_1) + \frac{1}{5}P(C_2) + \frac{1}{5}P(C_3) = \frac{1}{5}.$$

Therefore, $P(A)$ must be inside $\left(\frac{1}{5}, \frac{1}{3}\right)$.

You may check that any value of $P(A)$ in the range $\boxed{\left(\frac{1}{5}, \frac{1}{3}\right)}$ can be obtained by first setting the value of $P(C_2)$ to be close to 0 and varying the value of $P(C_1)$ from 0 to 1.

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HW Solution 5 — Due: Sep 25, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. Series Circuit: The circuit in Figure 5.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

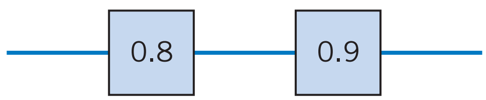

Figure 5.1: Circuit for Problem 1

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Problem 2. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.

Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Problem 3. You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random and tossed, falling heads up. How likely is it that it is the fair one? [Capinski and Zastawniak, 2003, Q7.28]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the probability $P(F)$ of selecting the fair coin is $P(F) = \frac{1}{2}$. For fair coin, the conditional probability $P(H|F)$ of heads is $\frac{1}{2}$ For the unfair coin, $P(U) = 1 - P(F) = \frac{1}{2}$ and $P(H|U) = \frac{1}{3}$.

By the Bayes' formula, the probability that the fair coin has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{3}} = \frac{1}{1 + \frac{2}{3}} = \boxed{\frac{3}{5}}.$$

Problem 4. You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails $1-p$. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins? [Capinski and Zastawniak, 2003, Q7.29]

Solution: Let F, U, and H be the events that "the selected coin is fair", "the selected coin is unfair", and "the coin lands heads up", respectively. We are given that

$$P(F) = \frac{2}{3}, \quad P(U) = \frac{1}{3}, \quad P(H|F) = \frac{1}{2}, P(H|U) = p.$$

By Bayes' theorem, the probability that one of the fair coins has been selected given that it lands heads up is

$$P(F|H) = \frac{P(H|F)P(F)}{P(H|F)P(F) + P(H|U)P(U)} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + p \times \frac{1}{3}} = \boxed{\frac{1}{1+p}}.$$

Alternative Solution: Let F_1, F_2, U and H be the events that "the selected coin is the first fair coin", "the selected coin is the second fair coin", "the selected coin is unfair", and "the coin lands heads up", respectively.

Because the coin is selected at random, the events $F1, F2$, and U are equally likely:

$$P(F_1) = P(F_2) = P(U) = \frac{1}{3}.$$

For fair coins, the conditional probability of heads is $\frac{1}{2}$ and for the unfair coin, the conditional probability of heads is p:

$$P(H|F_1) = P(H|F_2) = \frac{1}{2}, \quad P(H|U) = p.$$

The probability that one of the fair coins has been selected given that it lands heads up is $P(F_1 \cup F_2|H)$. Now F_1 and F_2 are disjoint events. Therefore,

$$P(F_1 \cup F_2|H) = P(F_1|H) + P(F_2|H).$$

By Bayes' theorem,

$$P(F_1|H) = \frac{P(H|F_1)P(F_1)}{P(H)} \quad \text{and} \quad P(F_2|H) = \frac{P(H|F_2)P(F_2)}{P(H)}.$$

Therefore,

$$P(F_1 \cup F_2|H) = \frac{P(H|F_1)P(F_1)}{P(H)} + \frac{P(H|F_2)P(F_2)}{P(H)} = \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H)}.$$

Note that the collection of three events F_1, F_2, and U partitions the sample space. Therefore, by the total probability theorem,

$$P(H) = P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U).$$

Plugging the above expression of $P(H)$ into our expression for $P(F_1 \cup F_2|H)$ gives

$$\begin{aligned} P(F_1 \cup F_2|H) &= \frac{P(H|F_1)P(F_1) + P(H|F_2)P(F_2)}{P(H|F_1)P(F_1) + P(H|F_2)P(F_2) + P(H|U)P(U)} \\ &= \frac{\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{3} + p\times\frac{1}{3}} = \boxed{\frac{1}{1+p}}. \end{aligned}$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. In his book *Chances: Risk and Odds in Everyday Life*, James Burke says that there is a 72% chance a polygraph test (lie detector test) will catch a person who is, in fact, lying. Furthermore, there is approximately a 7% chance that the polygraph will falsely accuse someone of lying. [Brase and Brase, 2011, Q4.2.26]

(a) If the polygraph indicated that 30% of the questions were answered with lies, what would you estimate for the actual percentage of lies in the answers?

(b) If the polygraph indicated that 70% of the questions were answered with lies, what would you estimate for the actual percentage of lies?

Solution: Let AT and AL be the events that "the person actually answers the truth" and "the person actually answers with lie", respectively. Also, let PT and PL be the events that "the polygraph indicates that the answer is the truth" and "the polygraph indicates that the answer is a lie", respectively.

We know, from the provided information, that $P(PL|AL) = 0.72$ and that $P(PL|AT) = 0.07$.

Applying the total probability theorem, we have

$$\begin{aligned} P(PL) &= P(PL|AL)P(AL) + P(PL|AT)P(AT) \\ &= P(PL|AL)P(AL) + P(PL|AT)(1 - P(AL)). \end{aligned}$$

Solving for $P(AL)$, we have

$$P(AL) = \frac{P(PL) - P(PL|AT)}{P(PL|AL) - P(PL|AT)} = \frac{P(PL) - 0.07}{0.72 - 0.07} = \frac{P(PL) - 0.07}{0.65}.$$

(a) Plugging in $P(PL) = 0.3$, we have $P(AL) = \boxed{0.3538}$.

(b) Plugging in $P(PL) = 0.7$, we have $P(AL) = \boxed{0.9692}$.

Problem 6. Software to detect fraud in consumer phone cards tracks the number of metropolitan areas where calls originate each day. It is found that 1% of the legitimate users originate calls from two or more metropolitan areas in a single day. However, 30% of fraudulent users originate calls from two or more metropolitan areas in a single day. The proportion of fraudulent users is 0.01%. If the same user originates calls from two or more metropolitan areas in a single day, what is the probability that the user is fraudulent? [Montgomery and Runger, 2010, Q2-144]

Solution: Let F denote the event of fraudulent user and let M denote the event of originating calls from multiple (two or more) metropolitan areas in a day. Then,

$$\begin{aligned} P(F|M) &= \frac{P(M|F)P(F)}{P(M|F)P(F) + P(M|F^c)P(F^c)} = \frac{1}{1 + \frac{P(M|F^c)}{P(M|F)} \times \frac{P(F^c)}{P(F)}} \\ &= \frac{1}{1 + \frac{\frac{1}{100}}{\frac{30}{100}} \times \frac{\frac{9999}{10^4}}{\frac{1}{10^4}}} = \frac{1}{1 + \frac{9999}{30}} = \frac{30}{30 + 9999} = \frac{30}{10029} \approx \boxed{0.0030}. \end{aligned}$$

ECS 315: Probability and Random Processes **2018/1**

HW Solution 6 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{2, 3\}$.

  (i) Determine whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

  (ii) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2, 3, 4\}$, $A_2 = A_3 = \{4, 5, 6\}$.

  (i) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (ii) Check whether $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i \neq j$.

  (iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

  (i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

  (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P(A_1 \cap A_2 \cap A_3) = 0$, which is *not* the same as $P(A_1)P(A_2)P(A_3)$.

  (iii) No. Although the three conditions for pairwise independence are satisfied, the last (product) condition for independence among three events is not.

Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

  (i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
  $P(A_1)P(A_2)P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
  Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
  $P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
  $P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
  Hence, $P(A_i \cap A_j) \neq P(A_i)P(A_j)$ for all $i \neq j$.

(iii) No. TO be independent, the three events must satisfy four conditions. Here, only one is satisfied.

Remark: This counter-example shows that one product condition does not imply independence.

Problem 2 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Problem 3. The circuit in Figure 6.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-34]

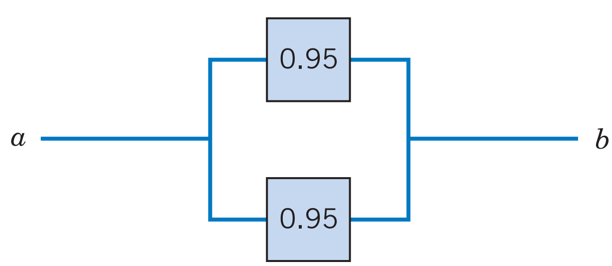

Figure 6.1: Circuit for Problem 3

Solution: Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. Therefore, the probability that the circuit operates is $P(T \cup B)$. Note that

$$P(T \cup B) = 1 - P((T \cup B)^c) = 1 - P(T^c \cap B^c).$$

We are told that $T^c \perp\!\!\!\perp B^c$. By their independence,

$$P(T^c \cap B^c) = P(T^c)P(B^c) = (1 - 0.95) \times (1 - 0.95) = 0.05^2 = 0.0025.$$

Therefore,

$$P(T \cup B) = 1 - P(T^c \cap B^c) = 1 - 0.0025 = \boxed{0.9975}.$$

Problem 4. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

(i) $P(A \cup B)$

(ii) $P(A \cup B^c)$

(iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]

Solution:

(a) By multiplication rule, there are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{6.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (6.1) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}}.$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (6.1) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}.}$$

For the event $A \cap B$, we replace "4" in (6.1) by "1" because we need red color and we replace "5" in the middle of (6.1) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) **Method 1**: $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^c \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Method 2: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = B^c \cup (A \cap B)$. Therefore,

$$P(A \cup B^c) = P(B^c) + P(A \cap B) = 1 - P(B) + P(A)P(B) = 1 - \frac{4}{5} + \frac{1}{4}\frac{4}{5} = \frac{2}{5}.$$

Method 3: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = A \cup (A^c \cap B^c)$. Therefore, $P(A \cup B^c) = P(A) + P(A^c \cap B^c)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B^c$. Hence,

$$P(A \cup B^c) = P(A) + P(A^c)P(B^c) = P(A) + (1 - P(A))(1 - P(B)) = \frac{1}{4} + \frac{3}{4}\frac{1}{5} = \frac{2}{5}.$$

(iii) **Method 1**: $P(A^c \cup B^c) = 1 - P\left((A^c \cup B^c)^c\right) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$

Method 2: From the Venn diagram, note that $A^c \cup B^c$ can be expressed as a disjoint union: $A^c \cup B^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$. Therefore,

$$P\left(A^c \cup B^c\right) = P\left(A^c \cap B\right) + P\left(A \cap B^c\right) + P\left(A^c \cap B^c\right).$$

Now, because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, and $A^c \perp\!\!\!\perp B^c$. Hence,

$$\begin{aligned} P\left(A^c \cup B^c\right) &= P\left(A^c\right)P\left(B\right) + P\left(A\right)P\left(B^c\right) + P\left(A^c\right)P\left(B^c\right) \\ &= \left(1 - P\left(A\right)\right)P\left(B\right) + P\left(A\right)\left(1 - P\left(B\right)\right) + \left(1 - P\left(A\right)\right)\left(1 - P\left(B\right)\right) \\ &= \frac{3}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{1}{5} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that

$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence

$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have

$$P(A^c \cap B) = P(B \cap A^c) = P(B)\,P(A^c)\,.$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)\,,$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))\,(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 6. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)\,P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2 - p}}.$$

Problem 7. The circuit in Figure 6.2 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

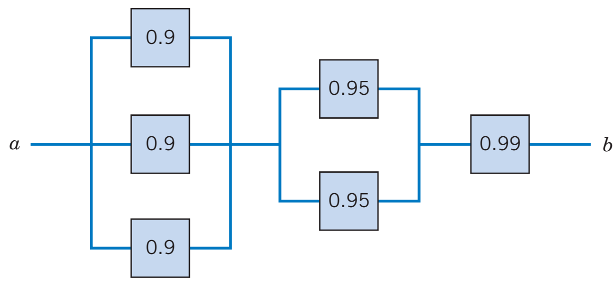

Figure 6.2: Circuit for Problem 7

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 3,

$$P(L) = 1 - (1 - 0.9)^3 = 1 - 0.1^3 = 0.999.$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem **??**, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$

Problem 8. An article in the British Medical Journal ["Comparison of Treatment of Renal Calculi by Operative Surgery, Percutaneous Nephrolithotomy, and Extracorporeal Shock Wave Lithotripsy" (1986, Vol. 82, pp. 879892)] provided the following discussion of success rates in kidney stone removals. Open surgery (OS) had a success rate of 78% (273/350) while a newer method, percutaneous nephrolithotomy (PN), had a success rate of 83% (289/350). This newer method looked better, but the results changed when stone diameter was considered. For stones with diameters less than two centimeters, 93% (81/87) of cases of open surgery were successful compared with only 87% (234/270) of cases of PN. For stones greater than or equal to two centimeters, the success rates were 73% (192/263) and 69% (55/80) for open surgery and PN, respectively. Open surgery is better for both stone sizes, but less successful in total. In 1951, E. H. Simpson pointed out this apparent contradiction (known as Simpson's Paradox) but the hazard still persists today. Explain how open surgery can be better for both stone sizes but worse in total. [Montgomery and Runger, 2010, Q2-115]

Solution: First, let's recall the total probability theorem:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)\,P(B) + P(A|B^c)\,P(B^c). \end{aligned}$$

We can see that $P(A)$ does not depend only on $P(A|B)$ and $P(A|B^c)$. It also depends on $P(B)$ and $P(B^c)$. In the extreme case, we may imagine the case with $P(B) = 1$ in which $P(A) = P(A|B)$. At another extreme, we may imagine the case with $P(B) = 0$ in which $P(A) = P(A|B^c)$. Therefore, depending on the value of $P(B)$, the value of $P(A)$ can be anywhere between $P(A|B)$ and $P(A|B^c)$.

Now, let's consider events A_1, B_1, A_2, and B_2. Let $P(A_1|B_1) = 0.93$ and $P(A_1|B_1^c) = 0.73$. Therefore, $P(A_1) \in [0.73, 0.93]$. On the other hand, let $P(A_2|B_2) = 0.87$ and $P(A_2|B_2^c) = 0.69$. Therefore, $P(A_2) \in [0.69, 0.87]$. With small value of $P(B_1)$, the value of $P(A_1)$ can be 0.78 which is closer to its lower end of the bound. With large value of $P(B_2)$, the value of $P(A_2)$ can be 0.83 which is closer to its upper end of the bound. Therefore, even though $P(A_1|B_1) > P(A_2|B_2) = 0.87$ and $P(A_1|B_1^c) > P(A_2|B_2^c)$, it is possible that $P(A_1) < P(A_2)$.

In the context of the paradox under consideration, note that the success rate of PN with small stones (87%) is higher than the success rate of OS with large stones (73%). Therefore, by having a lot of large stone cases to be tested under OS and also have a lot of small stone cases to be tested under PN, we can create a situation where the overall success rate of PN comes out to be better then the success rate of OS. This is exactly what happened in the study as shown in Table 6.1.

Problem 9. Show that

(a) $P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B)$.

(b) $P(B \cap C|A) = P(B|A)P(C|B \cap A)$

Solution:

(a) We can see directly from the definition of $P(B|A)$ that

$$P(A \cap B) = P(A)P(B|A).$$

Open surgery

	success	failure	sample size	sample percentage	conditional success rate
large stone	192	71	263	75%	73%
small stone	81	6	87	25%	93%
overall summary	273	77	350	100%	78%

PN

	success	failure	sample size	sample percentage	conditional success rate
large stone	55	25	80	23%	69%
small stone	234	36	270	77%	87%
overall summary	289	61	350	100%	83%

Table 6.1: Success rates in kidney stone removals.

Similarly, when we consider event $A \cap B$ and event C, we have

$$P(A \cap B \cap C) = P(A \cap B)\, P(C|A \cap B).$$

Combining the two equalities above, we have

$$P(A \cap B \cap C) = P(A) \times P(B|A) \times P(C|A \cap B).$$

(b) By definition,

$$P(B \cap C|A) = \frac{P(A \cap B \cap C)}{P(A)}.$$

Substitute $P(A \cap B \cap C)$ from part (a) to get

$$P(B \cap C|A) = \frac{P(A) \times P(B|A) \times P(C|A \cap B)}{P(A)} = P(B|A) \times P(C|A \cap B).$$