ECS 315: Probability and Random Processes **2017/1**

HW Solution 7 — Due: Oct 19, 4 PM

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Problem 1 (Quiz4, 2014)**.** Consider a random experiment in which you roll a 20-sided fair dice. We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega, \quad Y(\omega) = (\omega - 5)^2, \quad Z(\omega) = |\omega - 5| - 3$$

Evaluate the following probabilities:

(a) $P[X = 5]$

(b) $P[Y = 16]$

(c) $P[Y > 10]$

(d) $P[Z > 10]$

(e) $P[5 < Z < 10]$

Solution: In this question, $\Omega = \{1, 2, 3, \ldots, 20\}$ because the dice has 20 sides. All twenty outcomes are equally-likely because the dice is fair. So, the probability of each outcome is $\frac{1}{20}$:

$$P(\{\omega\}) = \frac{1}{20} \text{ for any } \omega \in \Omega.$$

(a) From $X(\omega) = \omega$, we have $X(\omega) = 5$ if and only if $\omega = 5$.

Therefore, $P[X = 5] = P(\{5\}) = \boxed{\frac{1}{20}}$.

(b) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) = 16$ if and only if $\omega = \pm 4 + 5 = 1$ or 9.

Therefore, $P[Y = 16] = P(\{1, 9\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(c) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) > 10$ if and only if $(\omega - 5)^2 > 10$. The values of ω that satisfy this condition are $1, 9, 10, 11, \ldots, 20$.

Therefore, $P[Y > 10] = P(\{1, 9, 10, 11, \ldots, 20\}) = \boxed{\frac{13}{20}}$.

(d) The values of ω that satisfy $|\omega - 5| - 3 > 10$ are 19 and 20.

Therefore, $P[Z > 10] = P(\{19, 20\}) = \frac{2}{20} = \boxed{\frac{1}{10}}$.

Calculations for Problem 1:

a) Find $P[X=5]$

$X(\omega)=5$ iff $\omega=5$

So, $P[X=5]=P(\{5\})=\frac{1}{20}$

b) Find $P[Y=16]$

$Y(\omega)=16$ iff $(\omega-5)^2=16$

$\omega-5=\pm 4$

$\omega=5\pm 4=1$ or 9

So, $P[Y=16]=P(\{1,9\})=\frac{2}{20}=\frac{1}{10}$

c) Find $P[Y>10]$

$Y(\omega)>10$ iff $(\omega-5)^2>10$

Here, we plug-in $\omega=1,2,\dots,20$ one-by-one and see that $\omega=1,9,10,11,\dots,20$ satisfy the condition.

So, $P[Y>10]=P(\{1,9,10,11,\dots,20\})=\frac{13}{20}$

ω	$(\omega-5)^2$	>10?
1	16	✓
2	9	✗
3	4	✗
4	1	✗
5	0	✗
6	1	✗
7	4	✗
8	9	✗
9	16	✓
10	25	✓
$\vdots$	increasing so, >10	$\vdots$

Alternatively, you may remember that $(\omega-5)^2>10$ iff $\omega-5>\sqrt{10}$ or $\omega-5<-\sqrt{10}$

$\Updownarrow$ $\omega>\underbrace{5+\sqrt{10}}_{=8.1623}$ $\Updownarrow$ $\omega=9,10,\dots,20$

$\Updownarrow$ $\omega<\underbrace{5-\sqrt{10}}_{1.8377}$ $\Updownarrow$ $\omega=1$

d) Find $P[Z>10]$

$Z(\omega)>10$ iff $|\omega-5|-3>10$

$|\omega-5|>13$

Here, we plug-in $\omega=1,2,\dots,20$ one-by-one and see that $\omega=19,20$ satisfy the condition.

So, $P[Z>10]=P(\{19,20\})=\frac{2}{20}=\frac{1}{10}$

ω	$\lvert\omega-5\rvert$	>13?	$8<\cdot<13$?
1	4	✗	✗
2	3	✗	✗
3	2	✗	✗
4	1	✗	✗
5	0	✗	✗
6	1	✗	✗
7	2	✗	✗
8	3	✗	✗
9	4	✗	✗
10	5	✗	✗
11	6	✗	✗
12	7	✗	✗
13	8	✗	✗
14	9	✗	✓
15	10	✗	✓
16	11	✗	✓
17	12	✗	✓
18	13	✗	✗
19	14	✓	✗
20	15	✓	✗

Alternatively, you may remember that $|\omega-5|>13$ iff $\omega-5>13$ or $\omega-5<-13$

$\Updownarrow$ $\omega>18$ $\Updownarrow$ $\omega=19,20$

$\Updownarrow$ $\omega<-8$ $\uparrow$ No ω satisfies this

e) Find $P[5<Z<10]$

$5<Z(\omega)<10$ iff $5<|\omega-5|-3<10$

$8<|\omega-5|<13$

Here, we plug-in $\omega=1,2,\dots,20$ one-by-one and see that $\omega=14,15,16,17$ satisfy the condition.

So, $P[5<Z<10]=P(\{14,15,16,17\})=\frac{4}{20}=\frac{1}{5}$

Alternatively, you may remember that $|\omega-5|>8$ iff $\omega-5>8$ or $\omega-5<-8$

$\Updownarrow$ $\omega>13$

$\Updownarrow$ $\omega<-3$ $\uparrow$ No ω satisfies this

$|\omega-5|<13$ iff $-13<\omega-5<13$

$\Updownarrow$ $-8<\omega<18$

Here, we need to satisfy both these condition. So, $13<\omega<18$

$\Updownarrow$ $\omega=14,15,16,17$

Alternatively, $|\omega-5|=\begin{cases}\omega-5 & \text{when } \omega\geq 5\\ -(\omega-5) & \text{when } \omega<5\end{cases}$

$\omega\geq 5$:

$8<\omega-5<13$

$13<\omega<18$

and $\Rightarrow$ $13<\omega<18$

$\omega<5$:

$8<-(\omega-5)<13$

$-13<\omega-5<-8$

$-8<\omega<-3$

$-8<\omega<-3$ $\Leftarrow$ and

$\Downarrow$

can't happen

(e) The values of ω that satisfy $5 < |\omega - 5| - 3 < 10$ are $14, 15, 16, 17$.

Therefore, $P[5 < Z < 10] = P(\{14, 15, 16, 17\}) = \frac{4}{20} = \boxed{\frac{1}{5}}$.

Problem 2. For each description of a random variable X below, indicate whether X is a **discrete** random variable.

(a) X is the number of websites visited by a randomly chosen software engineer in a day.

(b) X is the number of classes a randomly chosen student is taking.

(c) X is the average height of the passengers on a randomly chosen bus.

(d) A game involves a circular spinner with eight sections labeled with numbers. X is the amount of time the spinner spins before coming to a rest.

(e) X is the thickness of the longest book in a randomly chosen library.

(f) X is the number of keys on a randomly chosen keyboard.

(g) X is the length of a randomly chosen person's arm.

Solution:We consider the number of possibilities for the values of X in each part. If the collection of possible values is countable (finite or countably infinite), then we conclude that the random variable is discrete. Otherwise, the random variable is not discrete. Therefore, the X defined in parts (a), (b), and (f) are discrete. The X defined in other parts are not discrete.