ECS 315: Probability and Random Processes **2017/1**

HW Solution 3 — Due: Sep 14, 4 PM

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Problem 1. If A, B, and C are disjoint events with $P(A) = 0.2$, $P(B) = 0.3$ and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$

(b) $P(A \cap B \cap C)$

(c) $P(A \cap B)$

(d) $P((A \cup B) \cap C)$

(e) $P(A^c \cap B^c \cap C^c)$

[Montgomery and Runger, 2010, Q2-75]

Solution:

(a) Because A, B, and C are disjoint, $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.3 + 0.2 + 0.4 = \boxed{0.9.}$

(b) Because A, B, and C are disjoint, $A \cap B \cap C = \emptyset$ and hence $P(A \cap B \cap C) = P(\emptyset) = \boxed{0}$.

(c) Because A and B are disjoint, $A \cap B = \emptyset$ and hence $P(A \cap B) = P(\emptyset) = \boxed{0}$.

(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. By the disjointness among A, B, and C, we have $(A \cap C) \cup (B \cap C) = \emptyset \cup \emptyset = \emptyset$. Therefore, $P((A \cup B) \cap C) = P(\emptyset) = \boxed{0}$.

(e) From $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$, we have $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C) = 1 - 0.9 = \boxed{0.1.}$

Problem 2. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:

(a) $P(A)$

(b) $P(B)$

(c) $P(A^c)$

(d) $P(A \cup B)$

(e) $P(A \cap B)$

[Montgomery and Runger, 2010, Q2-55]

Solution:

(a) Recall that the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Therefore,

$$\begin{aligned} P(A) &= P(\{a,b,c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) \\ &= 0.1 + 0.1 + 0.2 = \boxed{0.4.} \end{aligned}$$

(b) Again, the probability of a finite or countable event equals the sum of the probabilities of the outcomes in the event. Thus,

$$\begin{aligned} P(B) &= P(\{c,d,e\}) = P(\{c\}) + P(\{d\}) + P(\{e\}) \\ &= 0.2 + 0.4 + 0.2 = \boxed{0.8.} \end{aligned}$$

(c) Applying the complement rule, we have $P(A^c) = 1 - P(A) = 1 - 0.4 = \boxed{0.6.}$

(d) Note that $A \cup B = \Omega$. Hence, $P(A \cup B) = P(\Omega) = \boxed{1.}$

(e) $P(A \cap B) = P(\{c\}) = \boxed{0.2.}$

Problem 3. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.1}$$

(a) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

(b) What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

(c) Use the binomial theorem (3.3) to evaluate $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

Solution:

(a) The coefficient of $x^r y^{n-r}$ is $\binom{n}{r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} = \boxed{5,200,300}$.

(b) We start from the expansion of $(a+b)^n$. Then we set $a = 2x$ and $b = -3y$:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = \sum_{r=0}^{n} \binom{n}{r} (2x)^r (-3y)^{n-r} = . \sum_{r=0}^{n} \binom{n}{r} 2^r (-3)^{n-r} x^r y^{n-r}. \tag{3.2}$$

Therefore, the coefficient of $x^r y^{n-r}$ is $\binom{n}{r} 2^r (-3)^{n-r}$. Here, $n = 25$ and $r = 12$. So, the coefficient is $\binom{25}{12} 2^{12} (-3)^{13} = -\frac{25!}{12!13!} 2^{12} 3^{13} = \boxed{-33959763545702400}$.

(c) From (3.3), set $x = -1$ and $y = 1$, then we have $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (-1+1)^n = \boxed{0}$.

Problem 4. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[1] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

[1] This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. ***Binomial theorem***: For any positive integer n, we know that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}. \tag{3.3}$$

(a) Use the binomial theorem (3.3) to simplify the following sums

(i) $\sum\limits_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(ii) $\sum\limits_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r (1-x)^{n-r}$

(b) If we differentiate (3.3) with respect to x and then multiply by x, we have

$$\sum_{r=0}^{n} r \binom{n}{r} x^r y^{n-r} = nx(x+y)^{n-1}.$$

Use similar technique to simplify the sum $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r}$.

Solution:

(a) To deal with the sum involving only the even terms (or only the odd terms), we first use (3.3) to expand $(x+y)^n$ and $(x+(-y))^n$. When we add the expanded results, only the even terms in the sum are left. Similarly, when we find the difference between the two expanded results, only the the odd terms are left. More specifically,

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n + (y-x)^n\right), \text{ and}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \frac{1}{2}\left((x+y)^n - (y-x)^n\right).$$

If $x + y = 1$, then

$$\sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 + (1-2x)^n\right)}, \text{ and} \tag{3.4a}$$

$$\sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r} x^r y^{n-r} = \boxed{\frac{1}{2}\left(1 - (1-2x)^n\right)}. \tag{3.4b}$$

(b) $\sum_{r=0}^{n} r^2 \binom{n}{r} x^r y^{n-r} = \boxed{nx\left(x(n-1)(x+y)^{n-2} + (x+y)^{n-1}\right)}$.

Problem 6.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A \cap B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A \cup B)$. Hint: Smaller than the interval $[0, 1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A \cap B)$.

First, recall[2], from the lecture notes, that "$P(A \cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.5}$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{3.6}$$

Now, $P(A \cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \le P(A \cup B) \le 1 \tag{3.7}$$

Combining (3.7) with (3.6) gives

$$P(A) + P(B) - 1 \le P(A \cap B) \le P(A) + P(B). \tag{3.8}$$

The second inequality in (3.8) is not useful because (3.5) gives a better[3] bound. So, we will replace the second inequality with (3.5):

$$P(A) + P(B) - 1 \le P(A \cap B) \le \min\{P(A), P(B)\}. \tag{3.9}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \le P(A \cap B) \le 1 \tag{3.10}$$

Combining (3.10) and (3.9), we have

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B), 1\}.$$

[2] Again, to see this, note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$.

[3] When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A) + P(B) - 1), 0\} \leq P(A \cap B) \leq \min\{P(A), P(B)\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6}, \frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

(b) We will try to derive general bounds for $P(A \cup B)$.

By monotonicity, because both A and B are subset of $A \cup B$, we must have

$$P(A \cup B) \geq \max\{P(A), P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A \cup B) \leq P(A) + P(B).$$

Therefore,

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq P(A) + P(B).$$

Being a probability, $P(A \cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A), P(B), 0\} \leq P(A \cup B) \leq \min\{(P(A) + P(B)), 1\}.$$

Note that number 0 is not needed in the minimization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq \min\{(P(A) + P(B)), 1\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$, we have

$$P(A \cup B) \in \boxed{\left[\frac{1}{2}, \frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.