ECS 315: Probability and Random Processes **2017/1**

HW Solution 12 — Due: Not Due

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Problem 1 (Yates and Goodman, 2005, Q3.3.4)**.** The pdf of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \le y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $\mathbb{E}[Y]$.

(b) Find $\operatorname{Var} Y$.

Solution:

(a) Recall that, for continuous random variable Y,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when y is outside of the interval $[0,2)$, $f_Y(y) = 0$ and hence does not affect the integration. We only need to integrate over $[0,2)$ in which $f_Y(y) = \frac{y}{2}$. Therefore,

$$\mathbb{E}Y = \int_0^2 y\left(\frac{y}{2}\right)dy = \int_0^2 \frac{y^2}{2}dy = \left.\frac{y^3}{2\times 3}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(b) The variance of any random variable Y (discrete or continuous) can be found from

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2$$

We have already calculate $\mathbb{E}Y$ in the previous part. So, now we need to calculate $\mathbb{E}[Y^2]$. Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y) dy.$$

Here, $g(y) = y^2$. Therefore,

$$\mathbb{E}\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy.$$

Again, in the integration, we can ignore the y whose $f_Y(y) = 0$:

$$\mathbb{E}\left[Y^2\right] = \int_0^2 y^2 \left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^3}{2} dy = \left.\frac{y^4}{2 \times 4}\right|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\operatorname{Var} Y = \mathbb{E}\left[Y^2\right] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

Problem 2 (Yates and Goodman, 2005, Q3.3.6)**.** The cdf of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144, & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $f_V(v)$?

(b) What is $\mathbb{E}[V]$?

(c) What is $\operatorname{Var}[V]$?

(d) What is $\mathbb{E}[V^3]$?

Solution: First, let's check whether V is a continuous random variable. This can be done easily by checking whether its cdf $F_V(v)$ is a continuous function. The cdf of V is defined using three expressions. Note that each expression is a continuous function. So, we only need to check whether there is/are any jump(s) at the boundaries: $v = 5$ and $v = 7$. Plugging $v = 5$ into $(v+5)^2/144$ gives 0 which matches the value of the expression for $v < -5$. Plugging $v = 7$ into $(v+5)^2/144$ gives 1 which matches the value of the expression for $v \ge 7$. SO, there is no discontinuity in $F_V(v)$. It is a continuous function and hence V itself is a continuous random variable.

(a) We can find the pdf $f_V(v)$ at almost all of the v by finding the derivative of the cdf $F_V(v)$:

$$f_V(v) = \frac{d}{dv} F_V(v) = \begin{cases} 0, & v < -5, \\ \frac{v+5}{72}, & -5 < v < 7, \\ 0, & v > 7. \end{cases}$$

Note that we still haven't specified $f_V(v)$ at $v = 5$ and $v = 7$. This is because the formula for $F_V(v)$ changes at those points and hence to actually find the derivatives, we would need to look at both the left and right derivatives at these points. The derivative may not even exist there. The good news is that we don't have to actually find them because $v = 5$ and $v = 7$ correspond to just two points on the pdf. Because V is a continuous random variable, we can "define" or "set " $f_V(v)$ to be any values there. In this case, for brevity of the expression, let's set the pdf to be 0 there. This gives

$$f_V(v) = \frac{d}{dv}F_V(v) = \boxed{\begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}}$$

(b) $\mathbb{E}[V] = \int\limits_{-\infty}^{\infty} v f_V(v)\, dv = \int\limits_{-5}^{7} v\left(\frac{v+5}{72}\right)dv = \frac{1}{72}\int\limits_{-5}^{7} v^2 + 5v dv = \boxed{3}$.

(c) $\mathbb{E}[V^2] = \int\limits_{-\infty}^{\infty} v^2 f_V(v)\, dv = \int\limits_{-5}^{7} v^2\left(\frac{v+5}{72}\right)dv = 17.$

Therefore, $\operatorname{Var} V = \mathbb{E}[V^2] - (\mathbb{E}[V])^2 = 17 - 9 = \boxed{8}$.

(d) $\mathbb{E}[V^3] = \int\limits_{-\infty}^{\infty} v^3 f_V(v)\, dv = \int\limits_{-5}^{7} v^3\left(\frac{v+5}{72}\right)dv = \boxed{\frac{431}{5} = 86.2}$.

Problem 3 (Yates and Goodman, 2005, Q3.4.5)**.** X is a continuous uniform RV on the interval $(-5, 5)$.

(a) What is its pdf $f_X(x)$?

(b) What is its cdf $F_X(x)$?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathbb{E}[X^5]$?

(e) What is $\mathbb{E}\left[e^X\right]$?

Solution: For a uniform random variable X on the interval (a, b), we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \le x \le b \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & x > b. \end{cases}$$

In this problem, we have $a = -5$ and $b = 5$.

(a) $f_X(x) = \boxed{\begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \le x \le 5 \end{cases}}$

(b) $F_X(x) = \boxed{\begin{cases} 0, & x < -5, \\ \frac{x+5}{10}, & a \le x \le b. \\ 1, & x > 5 \end{cases}}$

(c) $\mathbb{E}X = \int\limits_{-\infty}^{\infty} x f_X(x)\,dx = \int\limits_{-5}^{5} x \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^2}{2}\right|_{-5}^{5} = \frac{1}{20}\left(5^2 - (-5)^2\right) = \boxed{0}$.

In general,

$$\mathbb{E}X = \int\limits_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x dx = \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}X = \boxed{0}$.

(d) $\mathbb{E}[X^5] = \int\limits_{-\infty}^{\infty} x^5 f_X(x)\,dx = \int\limits_{-5}^{5} x^5 \times \frac{1}{10} dx = \frac{1}{10} \left.\frac{x^6}{6}\right|_{-5}^{5} = \frac{1}{60}\left(5^6 - (-5)^6\right) = \boxed{0}$.

In general,

$$\mathbb{E}\left[X^5\right] = \int\limits_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b x^5 dx = \frac{1}{b-a} \left.\frac{x^6}{6}\right|_a^b = \frac{1}{b-a} \frac{b^6 - a^6}{2}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}[X^5] = \boxed{0}$.

(e) In general,

$$\mathbb{E}\left[e^X\right] = \int\limits_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a} \int\limits_a^b e^x dx = \frac{1}{b-a} \left. e^x \right|_a^b = \frac{e^b - e^a}{b-a}.$$

With $a = -5$ and $b = 5$, we have $\mathbb{E}\left[e^X\right] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$.

Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi} 1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t + \theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

(b) Y is another function of Θ.

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1 + \theta) \times 5\cos(7t_2 + \theta)d\theta \\
&= \frac{25}{2\pi} \int_0^{2\pi} \cos(7t_1 + \theta) \times \cos(7t_2 + \theta)d\theta.
\end{aligned}$$

Recall[1] the cosine identity

$$\cos(a) \times \cos(b) = \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right).$$

Therefore,

$$\begin{aligned}
\mathbb{E}Y &= \frac{25}{4\pi} \int_0^{2\pi} \cos(14t + 2\theta) + \cos(7(t_1 - t_2))\, d\theta \\
&= \frac{25}{4\pi}\left(\int_0^{2\pi} \cos(14t + 2\theta)\, d\theta + \int_0^{2\pi} \cos(7(t_1 - t_2))\, d\theta\right).
\end{aligned}$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1 - t_2)) \int_0^{2\pi} d\theta = \frac{25}{4\pi}\cos(7(t_1 - t_2))\, 2\pi = \boxed{\frac{25}{2}\cos(7(t_1 - t_2))}.$$

[1] This identity could be derived easily via the Euler's identity:

$$\begin{aligned}
\cos(a) \times \cos(b) &= \frac{e^{ja} + e^{-ja}}{2} \times \frac{e^{jb} + e^{-jb}}{2} = \frac{1}{4}\left(e^{ja}e^{jb} + e^{-ja}e^{jb} + e^{ja}e^{-jb} + e^{-ja}e^{-jb}\right) \\
&= \frac{1}{2}\left(\frac{e^{ja}e^{jb} + e^{-ja}e^{-jb}}{2} + \frac{e^{-ja}e^{jb} + e^{ja}e^{-jb}}{2}\right) \\
&= \frac{1}{2}\left(\cos(a + b) + \cos(a - b)\right).
\end{aligned}$$

Problem 5. A random variable X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constant m and positive number σ. Furthermore, when a Gaussian random variable has $m = 0$ and $\sigma = 1$, we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by Φ and its values (or its complementary values $Q(\cdot) = 1 - \Phi(\cdot)$) are traditionally provided by a table.

Suppose Z is a standard Gaussian random variable.

(a) Use the Φ table to find the following probabilities:

  (i) $P[Z < 1.52]$
  (ii) $P[Z < -1.52]$
  (iii) $P[Z > 1.52]$
  (iv) $P[Z > -1.52]$
  (v) $P[-1.36 < Z < 1.52]$

(b) Use the Φ table to find the value of c that satisfies each of the following relation.

  (i) $P[Z > c] = 0.14$
  (ii) $P[-c < Z < c] = 0.95$

Solution:

(a)

  (i) $P[Z < 1.52] = \Phi(1.52) = \boxed{0.9357.}$
  (ii) $P[Z < -1.52] = \Phi(-1.52) = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$
  (iii) $P[Z > 1.52] = 1 - P[Z < 1.52] = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643.}$
  (iv) It is straightforward to see that the area of $P[Z > -1.52]$ is the same as $P[Z < 1.52] = \Phi(1.52)$. So, $P[Z > -1.52] = \boxed{0.9357.}$
  Alternatively, $P[Z > -1.52] = 1 - P[Z \le -1.52] = 1 - \Phi(-1.52) = 1 - (1 - \Phi(1.52)) = \Phi(1.52)$.
  (v) $P[-1.36 < Z < 1.52] = \Phi(1.52) - \Phi(-1.36) = \Phi(1.52) - (1 - \Phi(1.36)) = \Phi(1.52) + \Phi(1.36) - 1 = 0.9357 + 0.9131 - 1 = \boxed{0.8488.}$

(b)

(i) $P[Z > c] = 1 - P[Z \le c] = 1 - \Phi(c)$. So, we need $1 - \Phi(c) = 0.14$ or $\Phi(c) = 1 - 0.14 = 0.86$. In the Φ table, we do not have exactly 0.86, but we have 0.8599 and 0.8621. Because 0.86 is closer to 0.8599, we answer the value of c whose $\phi(c) = 0.8599$. Therefore, $c \approx \boxed{1.08.}$

(ii) $P[-c < Z < c] = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1$. So, we need $2\Phi(c) - 1 = 0.95$ or $\Phi(c) = 0.975$. From the Φ table, we have $c \approx \boxed{1.96.}$

Problem 6. The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is a $\mathcal{N}(85, 100)$ random variable.

Remark: Do not forget that, for our class, the second parameter in $\mathcal{N}(\cdot, \cdot)$ is the variance (not the standard deviation).

(a) Express the cdf of T in terms of the Φ function.

(b) Express each of the following probabilities in terms of the Φ function(s). Make sure that the arguments of the Φ functions are positive. (Positivity is required so that we can directly use the Φ/Q tables to evaluate the probabilities.)

(i) $P[T > 100]$

(ii) $P[T < 60]$

(iii) $P[70 \le T \le 100]$

(c) Express each of the probabilities in part (b) in terms of the Q function(s). Again, make sure that the arguments of the Q functions are positive.

(d) Evaluate each of the probabilities in part (b) using the Φ/Q tables.

(e) Observe that the Φ table ("Table 4" from the lecture) stops at $z = 2.99$ and the Q table ("Table 5" from the lecture) starts at $z = 3.00$. Why is it better to give a table for $Q(z)$ instead of $\Phi(z)$ when z is large?

Solution:

(a) Recall that when $X \sim \mathcal{N}(m, \sigma^2)$, $F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$. Here, $T \sim \mathcal{N}(85, 10^2)$. Therefore,

$$F_T(t) = \boxed{\Phi\left(\frac{t-85}{10}\right).}$$

(b)

(i) $P[T > 100] = 1 - P[T \le 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) = 1 - \Phi(1.5)$

(ii) $P[T < 60] = P[T \leq 60]$ because T is a continuous random variable and hence $P[T = 60] = 0$. Now, $P[T \leq 60] = F_T(60) = \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) = \boxed{1 - \Phi(2.5).}$ Note that, for the last equality, we use the fact that $\Phi(-z) = 1 - \Phi(z)$.

(iii)

$$P[70 \leq T \leq 100] = F_T(100) - F_T(70) = \Phi\left(\frac{100-85}{10}\right) - \Phi\left(\frac{70-85}{10}\right)$$
$$= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - (1 - \Phi(1.5)) = \boxed{2\Phi(1.5) - 1.}$$

(c) In this question, we use the fact that $Q(x) = 1 - \Phi(x)$.

(i) $1 - \Phi(1.5) = \boxed{Q(1.5).}$

(ii) $1 - \Phi(2.5) = \boxed{Q(2.5).}$

(iii) $2\Phi(1.5) - 1 = 2(1 - Q(1.5)) - 1 = 2 - 2Q(1.5) - 1 = \boxed{1 - 2Q(1.5).}$

(d)

(i) $1 - \Phi(1.5) = 1 - 0.9332 = \boxed{0.0668.}$

(ii) $1 - \Phi(2.5) = 1 - 0.99379 = \boxed{0.0062.}$

(iii) $2\Phi(1.5) - 1 = 2(0.9332) - 1 = \boxed{0.8664.}$

(e) When z is large, $\Phi(z)$ will start with 0.999... The first few significant digits will all be the same and hence not quite useful to be there.

Problem 7. Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

Solution: Let T be the time to failure (in hours). We are given that $T \sim \mathcal{E}(\lambda)$ where $\lambda = 3 \times 10^{-4}$. Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find $P[T > 10^4]$.

We shall first provide the general formula for the ccdf $P[T > t]$ when $t > 0$:

$$P[T > t] = \int_t^\infty f_T(\tau) d\tau = \int_t^\infty \lambda e^{-\lambda\tau} d\tau = -e^{-\lambda\tau}\Big|_t^\infty = e^{-\lambda t}. \tag{12.1}$$

Therefore,

$$P\left[T > 10^4\right] = e^{-3\times10^{-4}\times10^4} = \boxed{e^{-3} \approx 0.0498.}$$

(b) We start with $P[T \le 7000] = 1 - P[T > 7000]$. Next, we apply (12.1) to get

$$P[T \le 7000] = 1 - P[T > 7000] = 1 - e^{-3\times10^{-4}\times7000} = \boxed{1 - e^{-2.1} \approx 0.8775.}$$