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ECS 315: Probability and Random Processes **2017/1**

HW 11 — Due: Nov 23, 4 PM

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Instructions

(a) This assignment has 1 pages.

(b) (1 pt) Work and write your answers **directly on these provided sheets** (not on other blank sheet(s) of paper). Hard-copies are distributed in class.

(c) (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.

(d) (8 pt) Try to solve all problems.

(e) Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

Problem 1 (Yates and Goodman, 2005, Q3.2.1). The random variable X has probability density function

unknown constant

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Use the pdf to find the following quantities.

(a) the constant c Recall that any pdf should integrate to 1.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx\,dx = c\int_0^2 x\,dx = c\left.\frac{x^2}{2}\right|_0^2 = \underbrace{2c}_{\text{This should } = 1.}$$

Therefore, $c = \frac{1}{2}$.

(b) $P[0 \le X \le 1]$

$$= \int_0^1 f_X(x)\,dx = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{2}\left.\frac{x^2}{2}\right|_0^1 = \frac{1}{4}.$$

(c) $P[-1/2 \le X \le 1/2]$. $= \displaystyle\int_{-1/2}^{1/2} f_X(x)\,dx = \int_{0}^{1/2} \frac{1}{2}x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^{1/2} = \frac{1}{16}$.

$f_X(x) = 0$ on $[-\frac{1}{2}, 0)$

(d) the cdf $F_X(x)$.

For $x<0$, because $f_X(t)=0$ for $t<0$, $F_X(x) = \displaystyle\int_{-\infty}^{x} f_X(t)\,dt = 0$

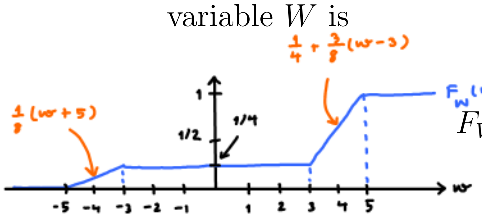
For $0 \le x \le 2$, $f_X(t) = \frac{t}{2}$ and $F_X(x) = \displaystyle\int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{t}{2}\,dt = \frac{t^2}{4}\Big|_0^x = \frac{x^2}{4}$.

At $x=2$, $F_X(2)=1$.

For $x>2$, $f_X(t)=0$. Therefore, $F_X(x) = \displaystyle\int_{-\infty}^{x} f_X(t)\,dt = \underbrace{\int_{-\infty}^{2} f_X(t)\,dt}_{F_X(2)=1} + \underbrace{\int_{2}^{x} \underbrace{f_X(t)}_{0}\,dt}_{0} = 1.$

$$F_X(x) = \begin{cases} 0, & x<0, \\ x^2/4, & 0 \le x \le 2, \\ 1, & \text{otherwise.} \end{cases}$$

Problem 2 (Modified from Yates and Goodman, 2005, Q3.1.3)**.** The CDF of a random variable W is



$$F_W(w) = \begin{cases} 0, & w < -5, \\ (w+5)/8, & -5 \le w < -3, \\ 1/4, & -3 \le w < 3, \\ 1/4 + 3(w-3)/8, & 3 \le w < 5, \\ 1, & w \ge 5. \end{cases}$$

Remark: It is possible to solve this problem by finding the pdf first. (You are asked to derive the pdf anyway in the enxt problem.) However, you should also make sure that you know how to calculate the probabilities directly from the cdf.

(a) Is W a continuous random variable?

From the plot above, we see that $F_W(w)$ is a continuous function.

Because its cdf is continuous, we conclude that W is a continuous RV.

(b) What is $P[W \le 4]$?

$P[W \le 4] = F_W(4) = \frac{1}{4} + \frac{3}{8}(4-3) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \approx 0.625$

by definition of cdf

(c) What is $P[-2 < W \le 2]$?

$P[-2 < W \le 2] = F_W(2) - F_W(-2) = \frac{1}{4} - \frac{1}{4} = 0$

For continuous RV, $P[a \le X \le b] = F_X(b) - F_X(a)$

(d) What is $P[W > 0]$?

$P[W > 0] = 1 - P[W \le 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(A) = 1 - P(A^c)$

(e) What is the value of a such that $P[W \le a] = 1/2$?

$P[W \le a] = F_W(a)$. From the plot above, we know that to have $F_W(a) = \frac{1}{2}$, the value of a must be in the interval $(3,5)$.

In this interval, $F_W(a) = \frac{1}{4} + \frac{3}{8}(a-3)$.

So, we solve for "a" that satisfies $\frac{1}{4} + \frac{3}{8}(a-3) = \frac{1}{2}$ $\Rightarrow a = \frac{11}{3} \approx 3.67$

Problem 3 (Yates and Goodman, 2005, Q3.2.3). The CDF of random variable W is

$$F_W(w) = \begin{cases} 0, & w < -5, \\ (w+5)/8, & -5 \le w < -3, \\ 1/4, & -3 \le w < 3, \\ 1/4 + 3(w-3)/8, & 3 \le w < 5, \\ 1, & w \ge 5. \end{cases}$$

Find its pdf $f_W(w)$.

Given a cdf, we can find the pdf by taking derivative.

As discussed in class, for the location(s) where derivative does not exist, we can choose to define the pdf to be any convenient value.

In this question, the cdf is given in the form of expressions on several intervals. It is then easy to find its derivative inside each of the intervals:

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 0, & w < -5, \\ 1/8, & -5 < w < -3, \\ 0, & -3 < w < 3, \\ 3/8, & 3 < w < 5, \\ 0, & 5 < w. \end{cases}$$

It should be clear from the plot of cdf in the previous problem that the derivative does not exist at $w = -5, -3, 3, 5$. We choose to assign $f_W(w) = 0$ at these points.

$$f_W(w) = \begin{cases} 1/8, & -5 < w < -3 \\ 3/8, & 3 < w < 5 \\ 0, & \text{otherwise} \end{cases}$$