

## HW Solution 7 — Due: Oct 19, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

**Problem 1** (Quiz4, 2014). Consider a random experiment in which you roll a 20-sided fair dice. We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega, \quad Y(\omega) = (\omega - 5)^2, \quad Z(\omega) = |\omega - 5| - 3$$

Evaluate the following probabilities:

- (a)  $P[X = 5]$
- (b)  $P[Y = 16]$
- (c)  $P[Y > 10]$
- (d)  $P[Z > 10]$
- (e)  $P[5 < Z < 10]$

**Solution:** In this question,  $\Omega = \{1, 2, 3, \dots, 20\}$  because the dice has 20 sides. All twenty outcomes are equally-likely because the dice is fair. So, the probability of each outcome is  $\frac{1}{20}$ .

$$P(\{\omega\}) = \frac{1}{20} \text{ for any } \omega \in \Omega.$$

- (a) From  $X(\omega) = \omega$ , we have  $X(\omega) = 5$  if and only if  $\omega = 5$ .

$$\text{Therefore, } P[X = 5] = P(\{5\}) = \boxed{\frac{1}{20}}.$$

- (b) From  $Y(\omega) = (\omega - 5)^2$ , we have  $Y(\omega) = 16$  if and only if  $\omega = \pm 4 + 5 = 1$  or  $9$ .

$$\text{Therefore, } P[Y = 16] = P(\{1, 9\}) = \frac{2}{20} = \boxed{\frac{1}{10}}.$$

- (c) From  $Y(\omega) = (\omega - 5)^2$ , we have  $Y(\omega) > 10$  if and only if  $(\omega - 5)^2 > 10$ . The values of  $\omega$  that satisfy this condition are  $1, 9, 10, 11, \dots, 20$ .

$$\text{Therefore, } P[Y > 10] = P(\{1, 9, 10, 11, \dots, 20\}) = \boxed{\frac{13}{20}}.$$

- (d) The values of  $\omega$  that satisfy  $|\omega - 5| - 3 > 10$  are  $19$  and  $20$ .

$$\text{Therefore, } P[Z > 10] = P(\{19, 20\}) = \frac{2}{20} = \boxed{\frac{1}{10}}.$$

Calculations for Problem 1:

a) Find  $P[X=5]$

$X(\omega) = 5$  iff  $\omega = 5$   
 So,  $P[X=5] = P(\{5\}) = \frac{1}{20}$

b) Find  $P[Y=16]$

$Y(\omega) = 16$  iff  $(\omega-5)^2 = 16$   
 $\omega - 5 = \pm 4$   
 $\omega = 5 \pm 4 = 1 \text{ or } 9$   
 So,  $P[Y=16] = P(\{1, 9\}) = \frac{2}{20} = \frac{1}{10}$

c) Find  $P[Y > 10]$

$Y(\omega) > 10$  iff  $(\omega-5)^2 > 10$

Here, we plug-in  $\omega = 1, 3, \dots, 20$  one-by-one and see that  $\omega = 1, 9, 10, 11, \dots, 20$  satisfy the condition.

So,  $P[Y > 10] = P(\{1, 9, 10, 11, \dots, 20\}) = \frac{12}{20}$

$\omega$	$(\omega-5)^2$	$> 10?$
1	16	✓
2	9	X
3	4	X
4	1	X
5	0	X
6	1	X
7	4	X
8	9	X
9	16	✓
10	25	✓
...	increasing	...

So,  $> 10$

Alternatively, you may remember that  $(\omega-5)^2 > 10$  iff  $\omega-5 > \sqrt{10}$  or  $\omega-5 < -\sqrt{10}$   
 $\omega > 5 + \sqrt{10} \approx 8.1623$   
 $\omega < 5 - \sqrt{10} \approx 1.8377$   
 $\omega = 9, 10, \dots, 20$        $\omega = 1$

d) Find  $P[Z > 10]$

$Z(\omega) > 10$  iff  $|\omega-5| - 3 > 10$   
 $|\omega-5| > 13$

Here, we plug-in  $\omega = 1, 3, \dots, 20$  one-by-one and see that  $\omega = 19, 20$  satisfy the condition.

So,  $P[Z > 10] = P(\{19, 20\}) = \frac{2}{20} = \frac{1}{10}$

Alternatively, you may remember that  $|\omega-5| > 13$  iff  $\omega-5 > 13$  or  $\omega-5 < -13$   
 $\omega > 18$   
 $\omega < -8$   
 $\omega = 19, 20$       No  $\omega$  satisfies this

$\omega$	$ \omega-5 $	$> 13?$	$< -13?$
1	4	X	X
2	3	X	X
3	2	X	X
4	1	X	X
5	0	X	X
6	1	X	X
7	2	X	X
8	3	X	X
9	4	X	X
10	5	X	X
11	6	X	X
12	7	X	X
13	8	X	X
14	9	X	X
15	10	X	X
16	11	X	X
17	12	X	X
18	13	X	X
19	14	✓	X
20	15	✓	X

e) Find  $P[5 < Z < 10]$

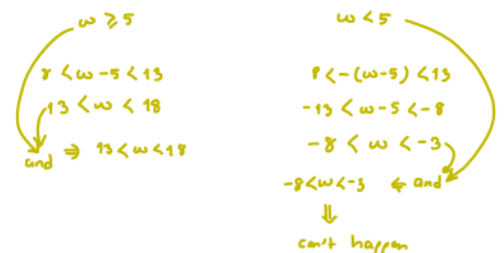
$5 < Z(\omega) < 10$  iff  $5 < |\omega-5| - 3 < 10$   
 $8 < |\omega-5| < 13$

Here, we plug-in  $\omega = 1, 3, \dots, 20$  one-by-one and see that  $\omega = 14, 15, 16, 17$  satisfy the condition.

So,  $P[5 < Z < 10] = P(\{14, 15, 16, 17\}) = \frac{4}{20} = \frac{1}{5}$

Alternatively,  $|\omega-5| = \begin{cases} \omega-5 & \text{when } \omega \geq 5 \\ -(\omega-5) & \text{when } \omega < 5 \end{cases}$

Alternatively, you may remember that  $|\omega-5| > 8$  iff  $\omega-5 > 8$  or  $\omega-5 < -8$   
 $\omega > 13$        $\omega < -3$   
 No  $\omega$  satisfies this  
 $|\omega-5| < 10$  iff  $-10 < \omega-5 < 10$   
 $-8 < \omega < 13$   
 Here,  $\omega$  need to satisfy both these condition. So,  $13 < \omega < 18$   
 $\omega = 14, 15, 16, 17$



(e) The values of  $\omega$  that satisfy  $5 < |\omega - 5| - 3 < 10$  are 14, 15, 16, 17.

$$\text{Therefore, } P[5 < Z < 10] = P(\{14, 15, 16, 17\}) = \frac{4}{20} = \boxed{\frac{1}{5}}.$$

**Problem 2.** For each description of a random variable  $X$  below, indicate whether  $X$  is a **discrete** random variable.

- (a)  $X$  is the number of websites visited by a randomly chosen software engineer in a day.
- (b)  $X$  is the number of classes a randomly chosen student is taking.
- (c)  $X$  is the average height of the passengers on a randomly chosen bus.
- (d) A game involves a circular spinner with eight sections labeled with numbers.  $X$  is the amount of time the spinner spins before coming to a rest.
- (e)  $X$  is the thickness of the longest book in a randomly chosen library.
- (f)  $X$  is the number of keys on a randomly chosen keyboard.
- (g)  $X$  is the length of a randomly chosen person's arm.

**Solution:** We consider the number of possibilities for the values of  $X$  in each part. If the collection of possible values is countable (finite or countably infinite), then we conclude that the random variable is discrete. Otherwise, the random variable is not discrete. Therefore, the  $X$  defined in parts (a), (b), and (f) are discrete. The  $X$  defined in other parts are not discrete.

## HW Solution 8 — Due: Oct 31, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

**Problem 1.** An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

- (a) Let the random variable  $X$  denote the number of parts that are correctly classified. Determine the probability mass function of  $X$ . [Montgomery and Runger, 2010, Q3-20]
- (b) Let the random variable  $Y$  denote the number of parts that are incorrectly classified. Determine the probability mass function of  $Y$ .

**Solution**<sup>1</sup>:

We will reexpress the problem in terms of Bernoulli trials so that we can use the results discussed in class. In this problem, we have three Bernoulli trials. Each trial deals with classification.

- (a) To find  $p_X(x)$ , first we find its support. Three parts are inspected here. Therefore,  $X$  can be 0, 1, 2, or 3. So, we need to find  $p_X(x) = P[X = x]$  when  $x = 0, 1, 2$  or 3. The pmf  $p_X(x)$  for other  $x$  values are all 0 because  $X$  cannot take the value of those  $x$ .

For each  $x \in \{0, 1, 2, 3\}$ ,  $p_X(x) = P[X = x]$  is simply the probability that exactly  $x$  parts are correctly classified. Note that, because we are interested in the correctly classified part, we define the “success” event for a trial to be the event that the part is classified correctly. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is  $p = 0.98$ . Under such interpretation (of “success”),  $p_X(x)$  is then the same as finding the probability of having exactly  $x$  successes in  $n = 3$  Bernoulli trials. We have seen in class that the probability of this is  $\binom{3}{x}p^x(1-p)^{3-x}$ . Plugging in  $p = 0.98$ , we have  $p_X(x) = \binom{3}{x}0.98^x(0.02)^{3-x}$  for  $x \in \{0, 1, 2, 3\}$ .

Combining the expression above with the cases for other  $x$  values, we then have

$$p_X(x) = \begin{cases} \binom{3}{x}0.98^x(0.02)^{3-x}, & x \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \quad (8.1)$$

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<sup>1</sup>The solution provided here assumes that we still haven’t reached the part of the course where binomial random variable is discussed. Therefore, the pmf is derived by relying on the concept of Bernoulli trials and the formula discussed back when we studied that topic.

In particular,  $p_X(0) = 8 \times 10^{-6}$ ,  $p_X(1) = 0.001176$ ,  $p_X(2) = 0.057624$ , and  $p_X(3) = 0.941192$ .

Remark: In fact, this  $X$  is a binomial random variable with  $n = 3$  and  $p = 0.98$ . In MATLAB, the probabilities above can be calculated via the command `binopdf(0:3,3,0.98)`.

- (b) **Method 1:** Similar analysis is performed on the random variable  $Y$ . The only difference here is that, now, we are interested in the number of parts that are *incorrectly* classified. Therefore, we will define the “success” event for a trial to be the event that the part is classified *incorrectly*. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is  $1 - p = 1 - 0.98 = 0.02$ . With this new probability of success, we have

$$p_Y(y) = \begin{cases} \binom{3}{y} 0.02^y (0.98)^{3-y}, & y \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \quad (8.2)$$

In particular,  $p_Y(0) = 0.941192$ ,  $p_Y(1) = 0.057624$ ,  $p_Y(2) = 0.001176$ , and  $p_Y(3) = 8 \times 10^{-6}$ .

Remark: In fact, this  $Y$  is a binomial random variable with  $n = 3$  and  $p = 0.02$ . In MATLAB, the probability values above can be calculated via the command `binopdf(0:3,3,0.02)`.

**Method 2:** Alternatively, note that there are three parts. If  $X$  of them are classified correctly, then the number of incorrectly classified parts is  $n - X$ , which is what we defined as  $Y$ . Therefore,  $Y = 3 - X$ . Hence,  $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$ .

**Problem 2.** Consider the sample space  $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$ . Suppose that  $P(A) = |A|/|\Omega|$  for any event  $A \subset \Omega$ . Define the random variable  $X(\omega) = \omega^2$ . Find the probability mass function of  $X$ .

**Solution:** The random variable maps the outcomes  $\omega = -2, -1, 0, 1, 2, 3, 4$  to numbers  $x = 4, 1, 0, 1, 4, 9, 16$ , respectively. Therefore,

$$\begin{aligned} p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7}, \\ p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7}, \\ p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7}, \\ p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and} \\ p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}. \end{aligned}$$

Combining the results above, we get the complete pmf:

$$p_X(x) = \begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}$$

**Problem 3.** Suppose  $X$  is a random variable whose pmf at  $x = 0, 1, 2, 3, 4$  is given by  $p_X(x) = \frac{2x+1}{25}$ .

Remark: Note that the statement above does not specify the value of the  $p_X(x)$  at the value of  $x$  that is not 0,1,2,3, or 4.

- (a) What is  $p_X(5)$ ?
- (b) Determine the following probabilities:
- $P[X = 4]$
  - $P[X \leq 1]$
  - $P[2 \leq X < 4]$
  - $P[X > -10]$

**Solution:**

- (a) First, we calculate

$$\sum_{x=0}^4 p_X(x) = \sum_{x=0}^4 \frac{2x+1}{25} = \frac{1+3+5+7+9}{25} = \frac{25}{25} = 1.$$

Therefore, there can't be any other  $x$  with  $p_X(x) > 0$ . At  $x = 5$ , we then conclude that  $p_X(5) = \boxed{0}$ . The same reasoning also implies that  $p_X(x) = 0$  at any  $x$  that is not 0,1,2,3, or 4.

- (b) Recall that, for discrete random variable  $X$ , the probability

$$P[\text{some condition(s) on } X]$$

can be calculated by adding  $p_X(x)$  for all  $x$  in the support of  $X$  that satisfies the given condition(s).

- $P[X = 4] = p_X(4) = \frac{2 \times 4 + 1}{25} = \boxed{\frac{9}{25}}$
- $P[X \leq 1] = p_X(0) + p_X(1) = \frac{2 \times 0 + 1}{25} + \frac{2 \times 1 + 1}{25} = \frac{1}{25} + \frac{3}{25} = \boxed{\frac{4}{25}}$

$$(iii) P[2 \leq X < 4] = p_X(2) + p_X(3) = \frac{2 \times 2 + 1}{25} + \frac{2 \times 3 + 1}{25} = \frac{5}{25} + \frac{7}{25} = \boxed{\frac{12}{25}}.$$

$$(iv) P[X > -10] = \boxed{1} \text{ because all the } x \text{ in the support of } X \text{ satisfies } x > -10.$$

**Problem 4.** The random variable  $V$  has pmf

$$p_V(v) = \begin{cases} cv^2, & v = 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the value of the constant  $c$ .
- (b) Find  $P[V \in \{u^2 : u = 1, 2, 3, \dots\}]$ .
- (c) Find the probability that  $V$  is an even number.
- (d) Find  $P[V > 2]$ .
- (e) Sketch  $p_V(v)$ .
- (f) Sketch  $F_V(v)$ . (Note that  $F_V(v) = P[V \leq v]$ .)

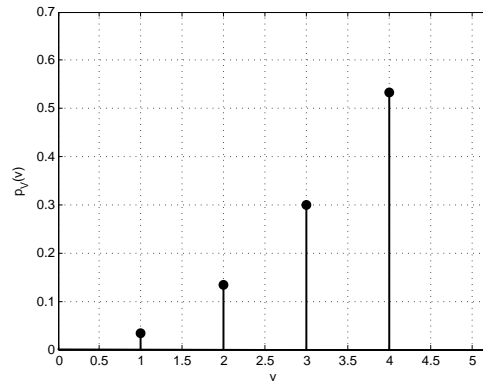
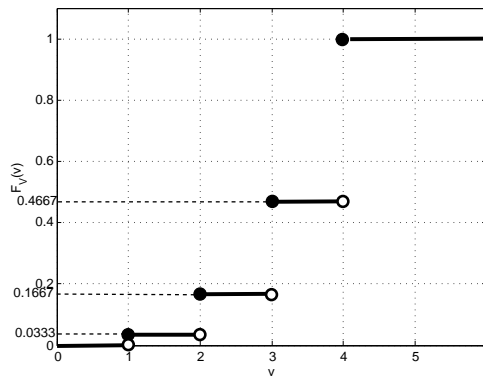
**Solution:** [Y&G, Q2.2.3]

- (a) We choose  $c$  so that the pmf sums to one:

$$\sum_v p_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1.$$

$$\text{Hence, } c = \boxed{1/30}.$$

- (b)  $P[V \in \{u^2 : u = 1, 2, 3, \dots\}] = P[V \in \{1, 4, 9, 16, 25\}] = p_V(1) + p_V(4) = c(1^2 + 4^2) = \boxed{17/30}$ .
- (c)  $P[V \text{ even}] = p_V(2) + p_V(4) = c(2^2 + 4^2) = 20/30 = \boxed{2/3}$ .
- (d)  $P[V > 2] = p_V(3) + p_V(4) = c(3^2 + 4^2) = 25/30 = \boxed{5/6}$ .
- (e) See Figure ?? for the sketch of  $p_V(v)$ :
- (f) See Figure ?? for the sketch of  $F_V(v)$ :

Figure 8.1: Sketch of  $p_V(v)$  for Question 4Figure 8.2: Sketch of  $F_V(v)$  for Question 4

**Problem 5.** The thickness of the wood paneling (in inches) that a customer orders is a random variable with the following cdf:

$$F_X(x) = \begin{cases} 0, & x < \frac{1}{8}, \\ 0.2, & \frac{1}{8} \leq x < \frac{1}{4}, \\ 0.9, & \frac{1}{4} \leq x < \frac{3}{8}, \\ 1 & x \geq \frac{3}{8}. \end{cases}$$

Determine the following probabilities:

- (a)  $P[X \leq 1/18]$
- (b)  $P[X \leq 1/4]$
- (c)  $P[X \leq 5/16]$
- (d)  $P[X > 1/4]$



(e)  $P[X \leq 1/2]$

[Montgomery and Runger, 2010, Q3-42]

**Solution:**

(a)  $P[X \leq 1/18] = F_X(1/18) = \boxed{0}$  because  $\frac{1}{18} < \frac{1}{8}$ .

(b)  $P[X \leq 1/4] = F_X(1/4) = \boxed{0.9}$ .

(c)  $P[X \leq 5/16] = F_X(5/16) = \boxed{0.9}$  because  $\frac{1}{4} < \frac{5}{16} < \frac{3}{8}$ .

(d)  $P[X > 1/4] = 1 - P[X \leq 1/4] = 1 - F_X(1/4) = 1 - 0.9 = \boxed{0.1}$ .

(e)  $P[X \leq 1/2] = F_X(1/2) = \boxed{1}$  because  $\frac{1}{2} > \frac{3}{8}$ .

Alternatively, we can also derive the pmf first and then calculate the probabilities.

## HW Solution 9 — Due: Nov 9, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

**Problem 1.** [F2013/1] For each of the following random variables, find  $P[1 < X \leq 2]$ .

(a)  $X \sim \text{Binomial}(3, 1/3)$

(b)  $X \sim \text{Poisson}(3)$

**Solution:**

(a) Because  $X \sim \text{Binomial}(3, 1/3)$ , we know that  $X$  can only take the values 0, 1, 2, 3. Only the value 2 satisfies the condition given. Therefore,  $P[1 < X \leq 2] = P[X = 2] = p_X(2)$ . Recall that the pmf for the binomial random variable is

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

for  $x = 0, 1, 2, 3, \dots, n$ . Here, it is given that  $n = 3$  and  $p = 1/3$ . Therefore,

$$p_X(2) = \binom{3}{2} \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right)^{3-2} = 3 \times \frac{1}{9} \times \frac{2}{3} = \boxed{\frac{2}{9}}.$$

(b) Because  $X \sim \text{Poisson}(3)$ , we know that  $X$  can take the values 0, 1, 2, 3,  $\dots$ . As in the previous part, only the value 2 satisfies the condition given. Therefore,  $P[1 < X \leq 2] = P[X = 2] = p_X(2)$ . Recall that the pmf for the Poisson random variable is

$$p_X(x) = e^{-\alpha} \frac{\alpha^x}{x!}$$

for  $x = 0, 1, 2, 3, \dots$ . Here, it is given that  $\alpha = 3$ . Therefore,

$$p_X(2) = e^{-3} \frac{3^2}{2!} = \boxed{\frac{9}{2} e^{-3} \approx 0.2240}.$$

**Problem 2.** Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of  $\lambda = 2$  customers per minute. Let  $M$  be the number of customers arriving between 9:00 and 9:05. What is the probability that  $M < 2$ ?

**Solution:** Here, we are given that  $M \sim \mathcal{P}(\alpha)$  where  $\alpha = \lambda T = 2 \times 5 = 10$ . Recall that, for  $M \sim \mathcal{P}(\alpha)$ , we have

$$P[M = m] = \begin{cases} e^{-\alpha} \frac{\alpha^m}{m!}, & m \in \{0, 1, 2, 3, \dots\} \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P[M < 2] &= P[M = 0] + P[M = 1] = e^{-\alpha} \frac{\alpha^0}{0!} + e^{-\alpha} \frac{\alpha^1}{1!} \\ &= e^{-\alpha} (1 + \alpha) = e^{-10} (1 + 10) = 11e^{-10} \approx 5 \times 10^{-4}. \end{aligned}$$

**Problem 3.** [M2011/1] The cdf of a random variable  $X$  is plotted in Figure 9.1.

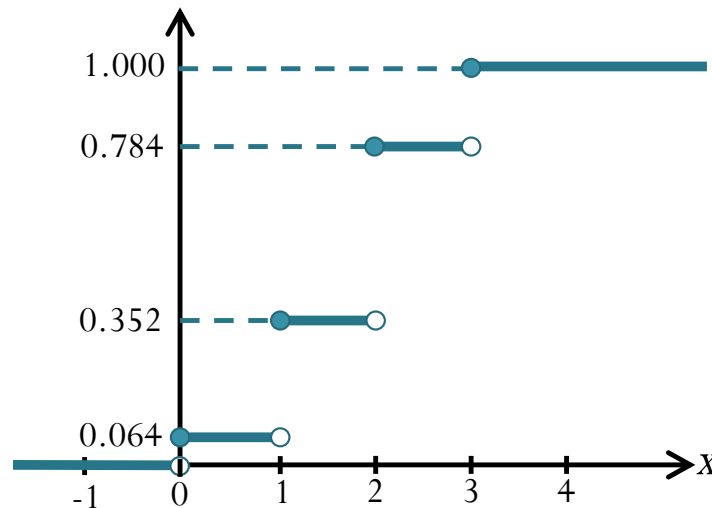


Figure 9.1: CDF of  $X$  for Problem 3

- Find the pmf  $p_X(x)$ .
- Find the family to which  $X$  belongs. (Uniform, Bernoulli, Binomial, Geometric, Poisson, etc.)

**Solution:**

- For discrete random variable,  $P[X = x]$  is the jump size at  $x$  on the cdf plot. In this problem, there are four jumps at 0, 1, 2, 3.
  - $P[X = 0] =$  the jump size at 0  $= 0.064 = \frac{64}{1000} = (4/10)^3 = (2/5)^3$ .
  - $P[X = 1] =$  the jump size at 1  $= 0.352 - 0.064 = 0.288$ .
  - $P[X = 2] =$  the jump size at 2  $= 0.784 - 0.352 = 0.432$ .
  - $P[X = 3] =$  the jump size at 3  $= 1 - 0.784 = 0.216 = (6/10)^3$ .

In conclusion,

$$p_X(x) = \begin{cases} 0.064, & x = 0, \\ 0.288, & x = 1, \\ 0.432, & x = 2, \\ 0.216, & x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

- (b) Among all the pmf that we discussed in class, only one can have support =  $\{0, 1, 2, 3\}$  with unequal probabilities. This is the binomial pmf. To check that it really is Binomial, recall that the pmf for binomial  $X$  is given by  $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$  for  $x = 0, 1, 2, \dots, n$ . Here,  $n = 3$ . Furthermore, observe that  $p_X(0) = (1-p)^n$ . By comparing  $p_X(0)$  with what we had in part (a), we have  $1-p = 2/5$  or  $p = 3/5$ . For  $x = 1, 2, 3$ , plugging in  $p = 3/5$  and  $n = 3$  into  $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$  gives the same values as what we had in part (a). So,  $X$  is a binomial RV.

**Problem 4.** When  $n$  is large, binomial distribution  $\text{Binomial}(n, p)$  becomes difficult to compute directly. In this question, we will consider an approximation when the value of  $p$  is close to 0. In such case, the binomial can be approximated<sup>1</sup> by the Poisson distribution with parameter  $\alpha = np$ . For this approximation to work, we will see in this exercise that  $n$  does not have to be very large and  $p$  does not need to be very small.

- (a) Let  $X \sim \text{Binomial}(12, 1/36)$ . (For example, roll two dice 12 times and let  $X$  be the number of times a double 6 appears.) Evaluate  $p_X(x)$  for  $x = 0, 1, 2$ .
- (b) Compare your answers part (a) with its Poisson approximation.

**Solution:**

- (a) For  $\text{Binomial}(n, p)$  random variable,

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, 2, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we are given that  $n = 12$  and  $p = \frac{1}{36}$ . Plugging in  $x = 0, 1, 2$ , we get 0.7132, 0.2445, 0.0384, respectively

<sup>1</sup>More specifically, suppose  $X_n$  has a binomial distribution with parameters  $n$  and  $p_n$ . If  $p_n \rightarrow 0$  and  $np_n \rightarrow \alpha$  as  $n \rightarrow \infty$ , then

$$P[X_n = k] \rightarrow e^{-\alpha} \frac{\alpha^k}{k!}.$$

- (b) A Poisson random variable with parameter  $\alpha = np$  can approximate a Binomial( $n, p$ ) random variable when  $n$  is large and  $p$  is small. Here, with  $n = 12$  and  $p = \frac{1}{36}$ , we have  $\alpha = 12 \times \frac{1}{36} = \frac{1}{3}$ . The Poisson pmf at  $x = 0, 1, 2$  is given by  $e^{-\alpha} \frac{\alpha^x}{x!} = e^{-1/3} \frac{(1/3)^x}{x!}$ . Plugging in  $x = 0, 1, 2$  gives  $[0.7165, 0.2388, 0.0398]$ , respectively.

Figure ?? compares the two pmfs. Note how close they are!

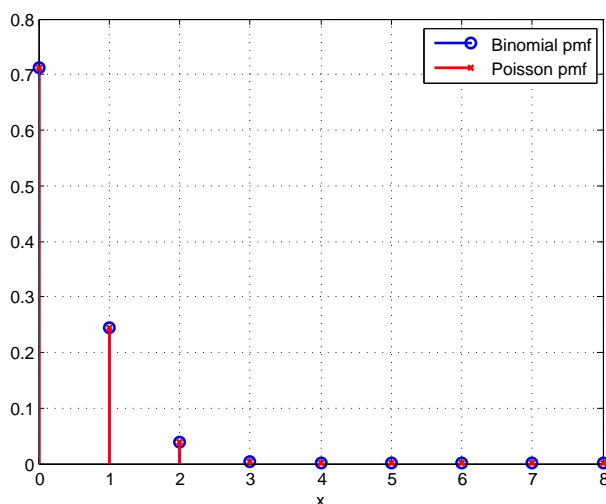


Figure 9.2: Poisson Approximation

**Problem 5.** You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson pmf. (For simplicity, exclude birthdays on February 29.) [Bertsekas and Tsitsiklis, 2008, Q2.2.2]

**Solution:** Let  $N$  be the number of guests that has the same birthday as you. We may think of the comparison of your birthday with each of the guests as a Bernoulli trial. Here, there are 500 guests and therefore we are considering  $n = 500$  trials. For each trial, the (success) probability that you have the same birthday as the corresponding guest is  $p = \frac{1}{365}$ . Then, this  $N \sim \text{Binomial}(n, p)$ .

(a) Binomial:  $P[N = 1] = np^1(1 - p)^{n-1} \approx 0.348$ .

(b) Poisson:  $P[N = 1] = e^{-np} \frac{(np)^1}{1!} \approx 0.348$ .

## Extra Questions

Here are some optional questions for those who want more practice.

**Problem 6.** A sample of a radioactive material emits particles at a rate of 0.7 per second. Assuming that these are emitted in accordance with a Poisson distribution, find the probability that in one second

- (a) exactly one is emitted,
- (b) more than three are emitted,
- (c) between one and four (inclusive) are emitted

[Applebaum, 2008, Q5.27].

**Solution:** Let  $X$  be the number of particles emitted during the one second under consideration. Then  $X \sim \mathcal{P}(\alpha)$  where  $\alpha = \lambda T = 0.7 \times 1 = 0.7$ .

$$(a) P[X = 1] = e^{-\alpha} \frac{\alpha^1}{1!} = \alpha e^{-\alpha} = 0.7e^{-0.7} \approx 0.3477.$$

$$(b) P[X > 3] = 1 - P[X \leq 3] = 1 - \sum_{k=0}^3 e^{-0.7} \frac{0.7^k}{k!} \approx 0.0058.$$

$$(c) P[1 \leq X \leq 4] = \sum_{k=1}^4 e^{-0.7} \frac{0.7^k}{k!} \approx 0.5026.$$

**Problem 7** (M2011/1). You are given an unfair coin with probability of obtaining a heads equal to  $1/3,000,000,000$ . You toss this coin  $6,000,000,000$  times. Let  $A$  be the event that you get “tails for all the tosses”. Let  $B$  be the event that you get “heads for all the tosses”.

- (a) Approximate  $P(A)$ .
- (b) Approximate  $P(A \cup B)$ .

**Solution:** Let  $N$  be the number of heads among the  $n$  tosses. Then,  $N \sim \mathcal{B}(n, p)$ . Here, we have small  $p = 1/3 \times 10^9$  and large  $n = 6 \times 10^9$ . So, we can apply Poisson approximation. In other words,  $\mathcal{B}(n, p)$  is well-approximated by  $\mathcal{P}(\alpha)$  where  $\alpha = np = 2$ .

$$(a) P(A) = P[N = 0] = e^{-2} \frac{2^0}{0!} = \frac{1}{e^2} \approx 0.1353.$$

$$(b) P(A \cup B) = P[N = 0] + P[N = n] = e^{-2} \frac{2^0}{0!} + e^{-2} \frac{2^{6 \times 10^9}}{(6 \times 10^9)!}. \text{ The second term is extremely small compared to the first one. Hence, } P(A \cup B) \text{ is approximately the same as } P(A).$$

## HW Solution 10 — Due: Nov 16, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

**Problem 1.** Consider a random variable  $X$  whose pmf is

$$p_X(x) = \begin{cases} 1/2, & x = -1, \\ 1/4, & x = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $Y = X^2$ .

- (a) Find  $\mathbb{E}X$ .
- (b) Find  $\mathbb{E}[X^2]$ .
- (c) Find  $\text{Var } X$ .
- (d) Find  $\sigma_X$ .
- (e) Find  $p_Y(y)$ .
- (f) Find  $\mathbb{E}Y$ .
- (g) Find  $\mathbb{E}[Y^2]$ .

**Solution:**

$$(a) \mathbb{E}X = \sum_x xp_X(x) = (-1) \times \frac{1}{2} + (0) \times \frac{1}{4} + (1) \times \frac{1}{4} = -\frac{1}{2} + \frac{1}{4} = \boxed{-\frac{1}{4}}.$$

$$(b) \mathbb{E}[X^2] = \sum_x x^2 p_X(x) = (-1)^2 \times \frac{1}{2} + (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}.$$

$$(c) \text{Var } X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{3}{4} - \left(-\frac{1}{4}\right)^2 = \frac{3}{4} - \frac{1}{16} = \boxed{\frac{11}{16}}.$$

$$(d) \sigma_X = \sqrt{\text{Var } X} = \boxed{\frac{\sqrt{11}}{4}}.$$

- (e) First, we build a table to see which values  $y$  of  $Y$  are possible from the values  $x$  of  $X$ :

$x$	$p_X(x)$	$y$
-1	1/2	$(-1)^2 = 1$
0	1/4	$(0)^2 = 0$
1	1/4	$(1)^2 = 1$

Therefore, the random variable  $Y$  can take two values: 0 and 1.  $p_Y(0) = p_X(0) = 1/4$ .  $p_Y(1) = p_X(-1) + p_X(1) = 1/2 + 1/4 = 3/4$ . Therefore,

$$p_Y(y) = \begin{cases} 1/4, & y = 0, \\ 3/4, & y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(f)  $\mathbb{E}Y = \sum_y y p_Y(y) = (0) \times \frac{1}{4} + (1) \times \frac{3}{4} = \boxed{\frac{3}{4}}$ . Alternatively, because  $Y = X^2$ , we automatically have  $\mathbb{E}[Y] = \mathbb{E}[X^2]$ . Therefore, we can simply use the answer from part (b).

(g)  $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = (0)^2 \times \frac{1}{4} + (1)^2 \times \frac{3}{4} = \boxed{\frac{3}{4}}$ . Alternatively,

$$\mathbb{E}[Y^2] = \mathbb{E}[X^4] = \sum_x x^4 p_X(x) = (-1)^4 \times \frac{1}{2} + (0)^4 \times \frac{1}{4} + (1)^4 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

**Problem 2.** For each of the following random variables, find  $\mathbb{E}X$  and  $\sigma_X$ .

(a)  $X \sim \text{Binomial}(3, 1/3)$

(b)  $X \sim \text{Poisson}(3)$

**Solution:**

(a) From the lecture notes, we know that when  $X \sim \text{Binomial}(n, p)$ , we have  $\mathbb{E}X = np$  and  $\text{Var} X = np(1-p)$ . Here,  $n = 3$  and  $p = 1/3$ . Therefore,  $\mathbb{E}X = 3 \times \frac{1}{3} = \boxed{1}$ . Also,

because  $\text{Var} X = 3 \left(\frac{1}{3}\right) \left(1 - \frac{1}{3}\right) = \frac{2}{3}$ , we have  $\sigma_X = \sqrt{\text{Var} X} = \boxed{\sqrt{\frac{2}{3}}}$ .

(b) From the lecture notes, we know that when  $X \sim \text{Poisson}(\alpha)$ , we have  $\mathbb{E}X = \alpha$  and  $\text{Var} X = \alpha$ . Here,  $\alpha = 3$ . Therefore,  $\mathbb{E}X = \boxed{3}$ . Also, because  $\text{Var} X = 3$ , we have  $\sigma_X = \boxed{\sqrt{3}}$ .



**Problem 3.** Suppose  $X$  is a uniform discrete random variable on  $\{-3, -2, -1, 0, 1, 2, 3, 4\}$ . Find

- (a)  $\mathbb{E}X$
- (b)  $\mathbb{E}[X^2]$
- (c)  $\text{Var } X$
- (d)  $\sigma_X$

**Solution:** All of the calculations in this question are simply plugging in numbers into appropriate formulas.

- (a)  $\mathbb{E}X = \boxed{0.5}$
- (b)  $\mathbb{E}[X^2] = \boxed{5.5}$
- (c)  $\text{Var } X = \boxed{5.25}$
- (d)  $\sigma_X = \boxed{2.2913}$

Alternatively, we can find a formula for the general case of uniform random variable  $X$  on the sets of integers from  $a$  to  $b$ . Note that there are  $n = b - a + 1$  values that the random variable can take. Hence, all of them has probability  $\frac{1}{n}$ .

- (a)  $\mathbb{E}X = \sum_{k=a}^b k \frac{1}{n} = \frac{1}{n} \sum_{k=a}^b k = \frac{1}{n} \times \frac{n(a+b)}{2} = \frac{a+b}{2}$ .
- (b) First, note that

$$\begin{aligned} \sum_{i=a}^b k(k-1) &= \sum_{k=a}^b k(k-1) \left( \frac{(k+1) - (k-2)}{3} \right) \\ &= \frac{1}{3} \left( \sum_{k=a}^b (k+1)k(k-1) - \sum_{k=a}^b k(k-1)(k-2) \right) \\ &= \frac{1}{3} ((b+1)b(b-1) - a(a-1)(a-2)) \end{aligned}$$

where the last equality comes from the fact that there are many terms in the first sum that is repeated in the second sum and hence many cancellations.

Now,

$$\begin{aligned} \sum_{k=a}^b k^2 &= \sum_{k=a}^b (k(k-1) + k) = \sum_{k=a}^b k(k-1) + \sum_{k=a}^b k \\ &= \frac{1}{3} ((b+1)b(b-1) - a(a-1)(a-2)) + \frac{n(a+b)}{2} \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=a}^b k^2 \frac{1}{n} &= \frac{1}{3n} ((b+1)b(b-1) - a(a-1)(a-2)) + \frac{a+b}{2} \\ &= \frac{1}{3} a^2 - \frac{1}{6} a + \frac{1}{3} ab + \frac{1}{6} b + \frac{1}{3} b^2\end{aligned}$$

$$(c) \text{ Var } X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = \frac{1}{12} (b-a)(b-a+2) = \frac{1}{12} (n-1)(n+1) = \frac{n^2-1}{12}.$$

$$(d) \sigma_X = \sqrt{\text{Var } X} = \sqrt{\frac{n^2-1}{12}}.$$

**Problem 4.** (Expectation + pmf + Gambling + Effect of miscalculation of probability) In the eighteenth century, a famous French mathematician Jean Le Rond d'Alembert, author of several works on probability, analyzed the toss of two coins. He reasoned that because this experiment has THREE outcomes, (the number of heads that turns up in those two tosses can be 0, 1, or 2), the chances of each must be 1 in 3. In other words, if we let  $N$  be the number of heads that shows up, Alembert would say that

$$p_N(n) = 1/3 \quad \text{for } N = 0, 1, 2. \quad (10.1)$$

[Mlodinow, 2008, p 50–51]

We know that Alembert's conclusion was *wrong*. His three outcomes are not equally likely and hence classical probability formula can not be applied directly. The key is to realize that there are FOUR outcomes which are equally likely. We should not consider 0, 1, or 2 heads as the possible outcomes. There are in fact four equally likely outcomes: (heads, heads), (heads, tails), (tails, heads), and (tails, tails). These are the 4 possibilities that make up the sample space. The actual pmf for  $N$  is

$$p_N(n) = \begin{cases} 1/4, & n = 0, 2, \\ 1/2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose you travel back in time and meet Alembert. You could make the following bet with Alembert to gain some easy money. The bet is that if the result of a toss of two coins contains exactly one head, then he would pay you \$150. Otherwise, you would pay him \$100.

Let  $R$  be Alembert's profit from this bet and  $Y$  be the your profit from this bet.

(a) Then,  $R = -150$  if you win and  $R = +100$  otherwise. Use Alembert's *miscalculated* probabilities from (10.1) to determine the pmf of  $R$  (from Alembert's belief).

(b) Use Alembert's *miscalculated* probabilities from (10.1) (or the corresponding (miscalculated) pmf found in part (a)) to calculate  $\mathbb{E}R$ , the expected profit for Alembert.

Remark: You should find that  $\mathbb{E}R > 0$  and hence Alembert will be quite happy to accept your bet.

(c) Use the *actual* probabilities, to determine the pmf of  $R$ .

(d) Use the *actual* pmf, to determine  $\mathbb{E}R$ .

Remark: You should find that  $\mathbb{E}R < 0$  and hence Alembert should not accept your bet if he calculates the probabilities correctly.

(e) Note that  $Y = +150$  if you win and  $Y = -100$  otherwise. Use the *actual* probabilities to determine the pmf of  $Y$ .

(f) Use the *actual* probabilities, to determine  $\mathbb{E}Y$ .

Remark: You should find that  $\mathbb{E}Y > 0$ . This is the amount of money that you expect to gain each time that you play with Alembert. Of course, Alembert, who still believes that his calculation is correct, will ask you to play this bet again and again believing that he will make profit in the long run.

By miscalculating probabilities, one can make wrong decisions (and lose a lot of money)!

**Solution:**

(a)  $P[R = -150] = P[N = 1]$  and  $P[R = +100] = P[N \neq 1] = P[N = 0] + P[N = 2]$ . So,

$$p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise.} \end{cases}$$

Using Alembert's *miscalculated* pmf,

$$p_R(r) = \begin{cases} 1/3, & r = -150, \\ 2/3, & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

(b) From  $p_R(r)$  in part (a), we have  $\mathbb{E}R = \sum_r p_R(r) = \frac{1}{3} \times (-150) + \frac{2}{3} \times 100 = \boxed{\frac{50}{3}} \approx 16.67$

(c) Again,

$$p_R(r) = \begin{cases} p_N(1), & r = -150, \\ p_N(0) + p_N(2), & r = +100, \\ 0, & \text{otherwise} \end{cases}$$

Using the actual pmf,

$$p_R(r) = \begin{cases} \frac{1}{2}, & r = -150, \\ \frac{1}{4} + \frac{1}{4}, & r = +100, \\ 0, & \text{otherwise} \end{cases} = \boxed{\begin{cases} \frac{1}{2}, & r = -150 \text{ or } +100, \\ 0, & \text{otherwise.} \end{cases}}$$

(d) From  $p_R(r)$  in part (c), we have  $\mathbb{E}R = \sum_r p_R(r)r = \frac{1}{2} \times (-150) + \frac{1}{2} \times 100 = \boxed{-25}$ .

(e) Observe that  $Y = -R$ . Hence, using the answer from part (c), we have

$$p_Y(y) = \boxed{\begin{cases} \frac{1}{2}, & y = +150 \text{ or } -100, \\ 0, & \text{otherwise.} \end{cases}}$$

(f) Observe that  $Y = -R$ . Hence,  $\mathbb{E}Y = -\mathbb{E}R$ . Using the actual probabilities,  $\mathbb{E}R = -25$  from part (d). Hence,  $\mathbb{E}Y = \boxed{+25}$ .

## Extra Questions

Here are some optional questions for those who want more practice.

**Problem 5.** A random variables  $X$  has support containing only two numbers. Its expected value is  $\mathbb{E}X = 5$ . Its variance is  $\text{Var } X = 3$ . Give an example of the pmf of such a random variable.

**Solution:** We first find  $\sigma_X = \sqrt{\text{Var } X} = \sqrt{3}$ . Recall that this is the average deviation from the mean. Because  $X$  takes only two values, we can make them at exactly  $\pm\sqrt{3}$  from the mean; that is

$$x_1 = 5 - \sqrt{3} \quad \text{and} \quad x_2 = 5 + \sqrt{3}.$$

In which case, we automatically have  $\mathbb{E}X = 5$  and  $\text{Var } X = 3$ . Hence, one example of such pmf is

$$p_X(x) = \boxed{\begin{cases} \frac{1}{2}, & x = 5 \pm \sqrt{3} \\ 0, & \text{otherwise} \end{cases}}$$

We can also try to find a general formula for  $x_1$  and  $x_2$ . If we let  $p = P[X = x_2]$ , then  $q = 1 - p = P[X = x_1]$ . Given  $p$ , the values of  $x_1$  and  $x_2$  must satisfy two conditions:  $\mathbb{E}X = m$  and  $\text{Var } X = \sigma^2$ . (In our case,  $m = 5$  and  $\sigma^2 = 3$ .) From  $\mathbb{E}X = m$ , we must have

$$x_1q + x_2p = m; \tag{10.2}$$

that is

$$x_1 = \frac{m}{q} - x_2 \frac{p}{q}.$$

From  $\text{Var } X = \sigma^2$ , we have  $\mathbb{E}[X^2] = \text{Var } X + \mathbb{E}X^2 = \sigma^2 + m^2$  and hence we must have

$$x_1^2 q + x_2^2 p = \sigma^2 + m^2. \quad (10.3)$$

Substituting  $x_1$  from (10.2) into (10.3), we have

$$x_2^2 p - 2x_2 m p + (p m^2 - q \sigma^2) = 0$$

whose solutions are

$$x_2 = \frac{2mp \pm \sqrt{4m^2 p^2 - 4p(pm^2 - q\sigma^2)}}{2p} = \frac{2mp \pm 2\sigma\sqrt{pq}}{2p} = m \pm \sigma\sqrt{\frac{q}{p}}.$$

Using (10.2), we have

$$x_1 = \frac{m}{q} - \left(m \pm \sigma\sqrt{\frac{q}{p}}\right) \frac{p}{q} = m \mp \sigma\sqrt{\frac{p}{q}}.$$

Therefore, for any given  $p$ , there are two pmfs:

$$p_X(x) = \begin{cases} 1-p, & x = m - \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m + \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise,} \end{cases}$$

or

$$p_X(x) = \begin{cases} 1-p, & x = m + \sigma\sqrt{\frac{p}{1-p}} \\ p, & x = m - \sigma\sqrt{\frac{1-p}{p}} \\ 0, & \text{otherwise.} \end{cases}$$

**Problem 6.** For each of the following families of random variable  $X$ , find the value(s) of  $x$  which maximize  $p_X(x)$ . (This can be interpreted as the “mode” of  $X$ .)

- (a)  $\mathcal{P}(\alpha)$
- (b) Binomial( $n, p$ )
- (c)  $\mathcal{G}_0(\beta)$
- (d)  $\mathcal{G}_1(\beta)$

Remark [Y&G, p. 66]:

- For statisticians, the mode is the most common number in the collection of observations. There are as many or more numbers with that value than any other value. If there are two or more numbers with this property, the collection of observations is called multimodal. In probability theory, a **mode** of random variable  $X$  is a number  $x_{\text{mode}}$  satisfying

$$p_X(x_{\text{mode}}) \geq p_X(x) \quad \text{for all } x.$$

- For statisticians, the median is a number in the middle of the set of numbers, in the sense that an equal number of members of the set are below the median and above the median. In probability theory, a median,  $X_{\text{median}}$ , of random variable  $X$  is a number that satisfies

$$P[X < X_{\text{median}}] = P[X > X_{\text{median}}].$$

- Neither the mode nor the median of a random variable  $X$  need be unique. A random variable can have several modes or medians.

**Solution:** We first note that when  $\alpha > 0$ ,  $p \in (0, 1)$ ,  $n \in \mathbb{N}$ , and  $\beta \in (0, 1)$ , the above pmf's will be strictly positive for some values of  $x$ . Hence, we can discard those  $x$  at which  $p_X(x) = 0$ . The remaining points are all integers. To compare them, we will evaluate  $\frac{p_X(i+1)}{p_X(i)}$ .

(a) For Poisson pmf, we have

$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{e^{-\alpha} \alpha^{i+1}}{(i+1)!}}{\frac{e^{-\alpha} \alpha^i}{i!}} = \frac{\alpha}{i+1}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$  if and only if  $i < \alpha - 1$ .
- $\frac{p_X(i+1)}{p_X(i)} = 1$  if and only if  $i = \alpha - 1$ .
- $\frac{p_X(i+1)}{p_X(i)} < 1$  if and only if  $i > \alpha - 1$ .

Let  $\tau = \alpha - 1$ . This implies that  $\tau$  is the place where things change. Moving from  $i$  to  $i + 1$ , the probability strictly increases if  $i < \tau$ . When  $i > \tau$ , the next probability value (at  $i + 1$ ) will decrease.

- (i) Suppose  $\alpha \in (0, 1)$ , then  $\alpha - 1 < 0$  and hence  $i > \alpha - 1$  for all  $i$ . (Note that  $i$  are nonnegative integers.) This implies that the pmf is a strictly decreasing function and hence the maximum occurs at the first  $i$  which is  $i = 0$ .

- (ii) Suppose  $\alpha \in \mathbb{N}$ . Then, the pmf will be strictly increasing until we reaches  $i = \alpha - 1$ . At which point, the next probability value is the same. Then, as we further increase  $i$ , the pmf is strictly decreasing. Therefore, the maximum occurs at  $\alpha - 1$  and  $\alpha$ .
- (iii) Suppose  $\alpha \notin \mathbb{N}$  and  $\alpha \geq 1$ . Then we will have have any  $i = \alpha - 1$ . The pmf will be strictly increasing where the last increase is from  $i = \lfloor \alpha - 1 \rfloor$  to  $i + 1 = \lfloor \alpha - 1 \rfloor + 1 = \lfloor \alpha \rfloor$ . After this, the pmf is strictly decreasing. Hence, the maximum occurs at  $\lfloor \alpha \rfloor$ .

To summarize,

$$\arg \max_x p_X(x) = \begin{cases} 0, & \alpha \in (0, 1), \\ \alpha - 1 \text{ and } \alpha, & \alpha \text{ is an integer,} \\ \lfloor \alpha \rfloor, & \alpha > 1 \text{ is not an integer.} \end{cases}$$

(b) For binomial pmf, we have

$$\frac{p_X(i+1)}{p_X(i)} = \frac{\frac{n!}{(i+1)!(n-i-1)!} p^{i+1} (1-p)^{n-i-1}}{\frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}} = \frac{(n-i)p}{(i+1)(1-p)}.$$

Notice that

- $\frac{p_X(i+1)}{p_X(i)} > 1$  if and only if  $i < np - 1 + p = (n+1)p - 1$ .
- $\frac{p_X(i+1)}{p_X(i)} = 1$  if and only if  $i = (n+1)p - 1$ .
- $\frac{p_X(i+1)}{p_X(i)} < 1$  if and only if  $i > (n+1)p - 1$ .

Let  $\tau = (n+1)p - 1$ . This implies that  $\tau$  is the place where things change. Moving from  $i$  to  $i + 1$ , the probability strictly increases if  $i < \tau$ . When  $i > \tau$ , the next probability value (at  $i + 1$ ) will decrease.

- (i) Suppose  $(n+1)p$  is an integer. The pmf will strictly increase as a function of  $i$ , and then stays at the same value at  $i = \tau = (n+1)p - 1$  and  $i + 1 = (n+1)p - 1 + 1 = (n+1)p$ . Then, it will strictly decrease. So, the maximum occurs at  $(n+1)p - 1$  and  $(n+1)p$ .
- (ii) Suppose  $(n+1)p$  is not an integer. Then, there will not be any  $i$  that is  $= \tau$ . Therefore, we only have the pmf strictly increases where the last increase occurs when we goes from  $i = \lfloor \tau \rfloor$  to  $i + 1 = \lfloor \tau \rfloor + 1$ . After this, the probability is strictly decreasing. Hence, the maximum is unique and occur at  $\lfloor \tau \rfloor + 1 = \lfloor (n+1)p - 1 \rfloor + 1 = \lfloor (n+1)p \rfloor$ .

To summarize,

$$\arg \max_x p_X(x) = \begin{cases} (n+1)p - 1 \text{ and } (n+1)p, & (n+1)p \text{ is an integer,} \\ \lfloor (n+1)p \rfloor, & (n+1)p \text{ is not an integer.} \end{cases}$$

- (c)  $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$ . Hence,  $p_X(i)$  is strictly decreasing. The maximum occurs at the smallest value of  $i$  which is  $\boxed{0}$ .
- (d)  $\frac{p_X(i+1)}{p_X(i)} = \beta < 1$ . Hence,  $p_X(i)$  is strictly decreasing. The maximum occurs at the smallest value of  $i$  which is  $\boxed{1}$ .

**Problem 7.** An article in Information Security Technical Report [“Malicious Software—Past, Present and Future” (2004, Vol. 9, pp. 618)] provided the data (shown in Figure 10.1) on the top ten malicious software instances for 2002. The clear leader in the number of registered incidences for the year 2002 was the Internet worm “Klez”. This virus was first detected on 26 October 2001, and it has held the top spot among malicious software for the longest period in the history of virology.

Place	Name	% Instances
1	I-Worm.Klez	61.22%
2	I-Worm.Lentin	20.52%
3	I-Worm.Tanatos	2.09%
4	I-Worm.BadtransII	1.31%
5	Macro.Word97.Thus	1.19%
6	I-Worm.Hybris	0.60%
7	I-Worm.Bridex	0.32%
8	I-Worm.Magistr	0.30%
9	Win95.CIH	0.27%
10	I-Worm.Sircam	0.24%

Figure 10.1: The 10 most widespread malicious programs for 2002 (Source—Kaspersky Labs).

Suppose that 20 malicious software instances are reported. Assume that the malicious sources can be assumed to be independent.

- (a) What is the probability that at least one instance is “Klez”?
- (b) What is the probability that three or more instances are “Klez”?



- (c) What are the expected value and standard deviation of the number of “Klez” instances among the 20 reported?

**Solution:** Let  $N$  be the number of instances (among the 20) that are “Klez”. Then,  $N \sim \text{binomial}(n, p)$  where  $n = 20$  and  $p = 0.6122$ .

- (a)  $P[N \geq 1] = 1 - P[N < 1] = 1 - P[N = 0] = 1 - p_N(0) = 1 - \binom{20}{0} \times 0.6122^0 \times 0.3878^{20} \approx 0.9999999941 \approx 1$ .

(b)

$$\begin{aligned} P[N \geq 3] &= 1 - P[N < 3] = 1 - (P[N = 0] + P[N = 1] + P[N = 2]) \\ &= 1 - \sum_{k=0}^2 \binom{20}{k} (0.6122)^k (0.3878)^{20-k} \approx 0.999997 \end{aligned}$$

- (c)  $\mathbb{E}N = np = 20 \times 0.6122 = 12.244$ .

$$\sigma_N = \sqrt{\text{Var } N} = \sqrt{np(1-p)} = \sqrt{20 \times 0.6122 \times 0.3878} \approx 2.179.$$

## ECS 315: Probability and Random Processes

2017/1

## HW 11 — Due: Nov 23, 4 PM

Lecturer: Prapun Suksompong, Ph.D.

**Instructions**

- This assignment has 1 pages.
- (1 pt) Work and write your answers directly on these provided sheets (not on other blank sheet(s) of paper). Hard-copies are distributed in class.
- (1 pt) Write your first name and the last three digits of your student ID on the upper-right corner of this page.
- (8 pt) Try to solve all problems.
- Write down all the steps that you have done to obtain your answers. You may not get full credit even when your answer is correct without showing how you get your answer.

**Problem 1** (Yates and Goodman, 2005, Q3.2.1). The random variable  $X$  has probability density function

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

unknown constant

Use the pdf to find the following quantities.

- (a) the constant  $c$  Recall that any pdf should integrate to 1.

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^2 cx dx = c \int_0^2 x dx = c \frac{x^2}{2} \Big|_0^2 = \frac{4c}{2} = 2c$$

This should = 1. Therefore,  $c = \frac{1}{2}$ .

- (b)  $P[0 \leq X \leq 1]$

$$= \int_0^1 f_X(x) dx = \int_0^1 \frac{1}{2}x dx = \frac{1}{2} \frac{x^2}{2} \Big|_0^1 = \frac{1}{4}$$

(c)  $P[-1/2 \leq X \leq 1/2] = \int_{-1/2}^{1/2} f_X(x) dx = \int_0^{1/2} \frac{1}{2}x dx = \frac{1}{2} \frac{x^2}{2} \Big|_0^{1/2} = \frac{1}{16}$ .

$f_X(x) = 0$  on  $[-1/2, 0)$

(d) the cdf  $F_X(x)$ .

For  $x < 0$ , because  $f_X(t) = 0$  for  $t < 0$ ,  $F_X(x) = \int_{-\infty}^x f_X(t) dt = 0$

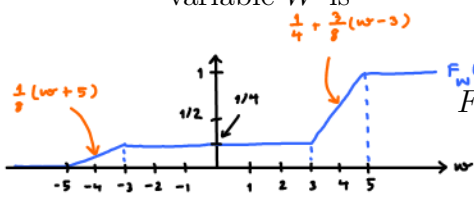
For  $0 \leq x \leq 2$ ,  $f_X(t) = \frac{t}{2}$  and  $F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_0^x \frac{t}{2} dt = \frac{t^2}{4} \Big|_0^x = \frac{x^2}{4}$ .

At  $x = 2$ ,  $F_X(2) = 1$ .

For  $x > 2$ ,  $f_X(t) = 0$ . Therefore,  $F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_{-\infty}^2 f_X(t) dt + \int_2^x f_X(t) dt = 1$ .

$F_X(x) = \begin{cases} 0, & x < 0, \\ x^2/4, & 0 \leq x \leq 2, \\ 1, & \text{otherwise.} \end{cases}$

**Problem 2** (Modified from Yates and Goodman, 2005, Q3.1.3). The CDF of a random variable  $W$  is



$$F_W(w) = \begin{cases} 0, & w < -5, \\ (w + 5)/8, & -5 \leq w < -3, \\ 1/4, & -3 \leq w < 3, \\ 1/4 + 3(w - 3)/8, & 3 \leq w < 5, \\ 1, & w \geq 5. \end{cases}$$

**Remark:** It is possible to solve this problem by finding the pdf first. (You are asked to derive the pdf anyway in the next problem.) However, you should also make sure that you know how to calculate the probabilities directly from the cdf.

(a) Is  $W$  a continuous random variable?

From the plot above, we see that  $F_W(w)$  is a continuous function.

Because its cdf is continuous, we conclude that  $W$  is a continuous RV.

(b) What is  $P[W \leq 4]$ ?

$$P[W \leq 4] = F_W(4) = \frac{1}{4} + \frac{3}{8}(4-3) = \frac{1}{4} + \frac{3}{8} = \frac{5}{8} \approx 0.625$$

by definition of cdf

(c) What is  $P[-2 < W \leq 2]$ ?

$$P[-2 < W \leq 2] = F_W(2) - F_W(-2) = \frac{1}{4} - \frac{1}{4} = 0$$

For continuous RV,  $P[a \leq X \leq b] = F_X(b) - F_X(a)$

(d) What is  $P[W > 0]$ ?

$$P[W > 0] = 1 - P[W \leq 0] = 1 - F_W(0) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A) = 1 - P(A^c)$$

(e) What is the value of  $a$  such that  $P[W \leq a] = 1/2$ ?

$P[W \leq a] = F_W(a)$ . From the plot above, we know that to have  $F_W(a) = \frac{1}{2}$ , the value of  $a$  must be in the interval  $(3, 5)$ .

In this interval,  $F_W(a) = \frac{1}{4} + \frac{3}{8}(a-3)$ .

So, we solve for "a" that satisfies  $\frac{1}{4} + \frac{3}{8}(a-3) = \frac{1}{2} \Rightarrow a = \frac{11}{3} \approx 3.67$

**Problem 3** (Yates and Goodman, 2005, Q3.2.3). The CDF of random variable  $W$  is

$$F_W(w) = \begin{cases} 0, & w < -5, \\ (w+5)/8, & -5 \leq w < -3, \\ 1/4, & -3 \leq w < 3, \\ 1/4 + 3(w-3)/8, & 3 \leq w < 5, \\ 1, & w \geq 5. \end{cases}$$

Find its pdf  $f_W(w)$ .

Given a cdf, we can find the pdf by taking derivative.

As discussed in class, for the location(s) where derivative does not exist, we can choose to define the pdf to be any convenient value.

In this question, the cdf is given in the form of expressions on several intervals. It is then easy to find its derivative inside each of the intervals:

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} 0, & w < -5, \\ 1/8, & -5 < w < -3, \\ 0, & -3 < w < 3, \\ 3/8, & 3 < w < 5, \\ 0, & 5 < w. \end{cases}$$

It should be clear from the plot of cdf in the previous problem that the derivative does not exist at  $w = -5, -3, 3, 5$ . We choose to assign

$f_W(w) = 0$  at these points.

$$f_W(w) = \begin{cases} 1/8, & -5 < w < -3 \\ 3/8, & 3 < w < 5 \\ 0, & \text{otherwise} \end{cases}$$

## HW Solution 12 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

**Problem 1** (Yates and Goodman, 2005, Q3.3.4). The pdf of random variable  $Y$  is

$$f_Y(y) = \begin{cases} y/2 & 0 \leq y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find  $\mathbb{E}[Y]$ .  
 (b) Find  $\text{Var } Y$ .

**Solution:**

- (a) Recall that, for continuous random variable  $Y$ ,

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) dy.$$

Note that when  $y$  is outside of the interval  $[0, 2)$ ,  $f_Y(y) = 0$  and hence does not affect the integration. We only need to integrate over  $[0, 2)$  in which  $f_Y(y) = \frac{y}{2}$ . Therefore,

$$\mathbb{E}Y = \int_0^2 y \left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^2}{2} dy = \frac{y^3}{2 \times 3} \Big|_0^2 = \boxed{\frac{4}{3}}.$$

- (b) The variance of any random variable  $Y$  (discrete or continuous) can be found from

$$\text{Var } Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2$$

We have already calculate  $\mathbb{E}Y$  in the previous part. So, now we need to calculate  $\mathbb{E}[Y^2]$ . Recall that, for continuous random variable,

$$\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y) dy.$$

Here,  $g(y) = y^2$ . Therefore,

$$\mathbb{E}[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy.$$

Again, in the integration, we can ignore the  $y$  whose  $f_Y(y) = 0$ :

$$\mathbb{E}[Y^2] = \int_0^2 y^2 \left(\frac{y}{2}\right) dy = \int_0^2 \frac{y^3}{2} dy = \frac{y^4}{2 \times 4} \Big|_0^2 = \boxed{2}.$$

Plugging this into the variance formula gives

$$\text{Var } Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \boxed{\frac{2}{9}}.$$

**Problem 2** (Yates and Goodman, 2005, Q3.3.6). The cdf of random variable  $V$  is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144, & -5 \leq v < 7, \\ 1 & v \geq 7. \end{cases}$$

- (a) What is  $f_V(v)$ ?
- (b) What is  $\mathbb{E}[V]$ ?
- (c) What is  $\text{Var}[V]$ ?
- (d) What is  $\mathbb{E}[V^3]$ ?

**Solution:** First, let's check whether  $V$  is a continuous random variable. This can be done easily by checking whether its cdf  $F_V(v)$  is a continuous function. The cdf of  $V$  is defined using three expressions. Note that each expression is a continuous function. So, we only need to check whether there is/are any jump(s) at the boundaries:  $v = 5$  and  $v = 7$ . Plugging  $v = 5$  into  $(v+5)^2/144$  gives 0 which matches the value of the expression for  $v < -5$ . Plugging  $v = 7$  into  $(v+5)^2/144$  gives 1 which matches the value of the expression for  $v \geq 7$ . SO, there is no discontinuity in  $F_V(v)$ . It is a continuous function and hence  $V$  itself is a continuous random variable.

- (a) We can find the pdf  $f_V(v)$  at almost all of the  $v$  by finding the derivative of the cdf  $F_V(v)$ :

$$f_V(v) = \frac{d}{dv} F_V(v) = \begin{cases} 0, & v < -5, \\ \frac{v+5}{72}, & -5 < v < 7, \\ 0, & v > 7. \end{cases}$$

Note that we still haven't specified  $f_V(v)$  at  $v = 5$  and  $v = 7$ . This is because the formula for  $F_V(v)$  changes at those points and hence to actually find the derivatives, we would need to look at both the left and right derivatives at these points. The derivative may not even exist there. The good news is that we don't have to actually

find them because  $v = 5$  and  $v = 7$  correspond to just two points on the pdf. Because  $V$  is a continuous random variable, we can “define” or “set”  $f_V(v)$  to be any values there. In this case, for brevity of the expression, let’s set the pdf to be 0 there. This gives

$$f_V(v) = \frac{d}{dv}F_V(v) = \begin{cases} \frac{v+5}{72}, & -5 < v < 7, \\ 0, & \text{otherwise.} \end{cases}$$

$$(b) \mathbb{E}[V] = \int_{-\infty}^{\infty} v f_V(v) dv = \int_{-5}^7 v \left(\frac{v+5}{72}\right) dv = \frac{1}{72} \int_{-5}^7 v^2 + 5v dv = \boxed{3}.$$

$$(c) \mathbb{E}[V^2] = \int_{-\infty}^{\infty} v^2 f_V(v) dv = \int_{-5}^7 v^2 \left(\frac{v+5}{72}\right) dv = 17.$$

$$\text{Therefore, } \text{Var } V = \mathbb{E}[V^2] - (\mathbb{E}[V])^2 = 17 - 9 = \boxed{8}.$$

$$(d) \mathbb{E}[V^3] = \int_{-\infty}^{\infty} v^3 f_V(v) dv = \int_{-5}^7 v^3 \left(\frac{v+5}{72}\right) dv = \boxed{\frac{431}{5} = 86.2}.$$

**Problem 3** (Yates and Goodman, 2005, Q3.4.5).  $X$  is a continuous uniform RV on the interval  $(-5, 5)$ .

(a) What is its pdf  $f_X(x)$ ?

(b) What is its cdf  $F_X(x)$ ?

(c) What is  $\mathbb{E}[X]$ ?

(d) What is  $\mathbb{E}[X^5]$ ?

(e) What is  $\mathbb{E}[e^X]$ ?

**Solution:** For a uniform random variable  $X$  on the interval  $(a, b)$ , we know that

$$f_X(x) = \begin{cases} 0, & x < a \text{ or } x > b, \\ \frac{1}{b-a}, & a \leq x \leq b \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b. \\ 1, & x > b. \end{cases}$$

In this problem, we have  $a = -5$  and  $b = 5$ .

$$(a) f_X(x) = \begin{cases} 0, & x < -5 \text{ or } x > 5, \\ \frac{1}{10}, & -5 \leq x \leq 5 \end{cases}$$

$$(b) F_X(x) = \begin{cases} 0, & x < -5, \\ \frac{x+5}{10}, & a \leq x \leq b. \\ 1, & x > 5 \end{cases}$$

$$(c) \mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-5}^5 x \times \frac{1}{10} dx = \frac{1}{10} \frac{x^2}{2} \Big|_{-5}^5 = \frac{1}{20} (5^2 - (-5)^2) = \boxed{0}.$$

In general,

$$\mathbb{E}X = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}.$$

With  $a = -5$  and  $b = 5$ , we have  $\mathbb{E}X = \boxed{0}$ .

$$(d) \mathbb{E}[X^5] = \int_{-\infty}^{\infty} x^5 f_X(x) dx = \int_{-5}^5 x^5 \times \frac{1}{10} dx = \frac{1}{10} \frac{x^6}{6} \Big|_{-5}^5 = \frac{1}{60} (5^6 - (-5)^6) = \boxed{0}.$$

In general,

$$\mathbb{E}[X^5] = \int_a^b x^5 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^5 dx = \frac{1}{b-a} \frac{x^6}{6} \Big|_a^b = \frac{1}{b-a} \frac{b^6 - a^6}{6}.$$

With  $a = -5$  and  $b = 5$ , we have  $\mathbb{E}[X^5] = \boxed{0}$ .

(e) In general,

$$\mathbb{E}[e^X] = \int_a^b e^x \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b e^x dx = \frac{1}{b-a} e^x \Big|_a^b = \frac{e^b - e^a}{b-a}.$$

With  $a = -5$  and  $b = 5$ , we have  $\mathbb{E}[e^X] = \boxed{\frac{e^5 - e^{-5}}{10}} \approx 14.84$ .

**Problem 4** (Randomly Phased Sinusoid). Suppose  $\Theta$  is a uniform random variable on the interval  $(0, 2\pi)$ .

(a) Consider another random variable  $X$  defined by

$$X = 5 \cos(7t + \Theta)$$

where  $t$  is some constant. Find  $\mathbb{E}[X]$ .



(b) Consider another random variable  $Y$  defined by

$$Y = 5 \cos(7t_1 + \Theta) \times 5 \cos(7t_2 + \Theta)$$

where  $t_1$  and  $t_2$  are some constants. Find  $\mathbb{E}[Y]$ .

**Solution:** First, because  $\Theta$  is a uniform random variable on the interval  $(0, 2\pi)$ , we know that  $f_{\Theta}(\theta) = \frac{1}{2\pi} 1_{(0, 2\pi)}(t)$ . Therefore, for “any” function  $g$ , we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_{\Theta}(\theta) d\theta.$$

(a)  $X$  is a function of  $\Theta$ .  $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5 \int_0^{2\pi} \frac{1}{2\pi} \cos(7t + \theta) d\theta$ . Now, we know that integration over a cycle of a sinusoid gives 0. So,  $\mathbb{E}[X] = \boxed{0}$ .

(b)  $Y$  is another function of  $\Theta$ .

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[5 \cos(7t_1 + \Theta) \times 5 \cos(7t_2 + \Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5 \cos(7t_1 + \theta) \times 5 \cos(7t_2 + \theta) d\theta \\ &= \frac{25}{2\pi} \int_0^{2\pi} \cos(7t_1 + \theta) \times \cos(7t_2 + \theta) d\theta. \end{aligned}$$

Recall<sup>1</sup> the cosine identity

$$\cos(a) \times \cos(b) = \frac{1}{2} (\cos(a + b) + \cos(a - b)).$$

Therefore,

$$\begin{aligned} \mathbb{E}Y &= \frac{25}{4\pi} \int_0^{2\pi} \cos(14t + 2\theta) + \cos(7(t_1 - t_2)) d\theta \\ &= \frac{25}{4\pi} (\int_0^{2\pi} \cos(14t + 2\theta) d\theta + \int_0^{2\pi} \cos(7(t_1 - t_2)) d\theta). \end{aligned}$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi} \cos(7(t_1 - t_2)) \int_0^{2\pi} d\theta = \frac{25}{4\pi} \cos(7(t_1 - t_2)) 2\pi = \boxed{\frac{25}{2} \cos(7(t_1 - t_2))}.$$

<sup>1</sup>This identity could be derived easily via the Euler’s identity:

$$\begin{aligned} \cos(a) \times \cos(b) &= \frac{e^{ja} + e^{-ja}}{2} \times \frac{e^{jb} + e^{-jb}}{2} = \frac{1}{4} (e^{ja}e^{jb} + e^{-ja}e^{jb} + e^{ja}e^{-jb} + e^{-ja}e^{-jb}) \\ &= \frac{1}{2} \left( \frac{e^{ja}e^{jb} + e^{-ja}e^{-jb}}{2} + \frac{e^{-ja}e^{jb} + e^{ja}e^{-jb}}{2} \right) \\ &= \frac{1}{2} (\cos(a + b) + \cos(a - b)). \end{aligned}$$

**Problem 5.** A random variable  $X$  is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2},$$

for some constant  $m$  and positive number  $\sigma$ . Furthermore, when a Gaussian random variable has  $m = 0$  and  $\sigma = 1$ , we say that it is a standard Gaussian random variable. There is no closed-form expression for the cdf of the standard Gaussian random variable. The cdf itself is denoted by  $\Phi$  and its values (or its complementary values  $Q(\cdot) = 1 - \Phi(\cdot)$ ) are traditionally provided by a table.

Suppose  $Z$  is a standard Gaussian random variable.

(a) Use the  $\Phi$  table to find the following probabilities:

- (i)  $P[Z < 1.52]$
- (ii)  $P[Z < -1.52]$
- (iii)  $P[Z > 1.52]$
- (iv)  $P[Z > -1.52]$
- (v)  $P[-1.36 < Z < 1.52]$

(b) Use the  $\Phi$  table to find the value of  $c$  that satisfies each of the following relation.

- (i)  $P[Z > c] = 0.14$
- (ii)  $P[-c < Z < c] = 0.95$

**Solution:**

(a)

- (i)  $P[Z < 1.52] = \Phi(1.52) = \boxed{0.9357}$ .
- (ii)  $P[Z < -1.52] = \Phi(-1.52) = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643}$ .
- (iii)  $P[Z > 1.52] = 1 - P[Z < 1.52] = 1 - \Phi(1.52) = 1 - 0.9357 = \boxed{0.0643}$ .
- (iv) It is straightforward to see that the area of  $P[Z > -1.52]$  is the same as  $P[Z < 1.52] = \Phi(1.52)$ . So,  $P[Z > -1.52] = \boxed{0.9357}$ .  
Alternatively,  $P[Z > -1.52] = 1 - P[Z \leq -1.52] = 1 - \Phi(-1.52) = 1 - (1 - \Phi(1.52)) = \Phi(1.52)$ .
- (v)  $P[-1.36 < Z < 1.52] = \Phi(1.52) - \Phi(-1.36) = \Phi(1.52) - (1 - \Phi(1.36)) = \Phi(1.52) + \Phi(1.36) - 1 = 0.9357 + 0.9131 - 1 = \boxed{0.8488}$ .

(b)

- (i)  $P[Z > c] = 1 - P[Z \leq c] = 1 - \Phi(c)$ . So, we need  $1 - \Phi(c) = 0.14$  or  $\Phi(c) = 1 - 0.14 = 0.86$ . In the  $\Phi$  table, we do not have exactly 0.86, but we have 0.8599 and 0.8621. Because 0.86 is closer to 0.8599, we answer the value of  $c$  whose  $\phi(c) = 0.8599$ . Therefore,  $c \approx \boxed{1.08}$ .
- (ii)  $P[-c < Z < c] = \Phi(c) - \Phi(-c) = \Phi(c) - (1 - \Phi(c)) = 2\Phi(c) - 1$ . So, we need  $2\Phi(c) - 1 = 0.95$  or  $\Phi(c) = 0.975$ . From the  $\Phi$  table, we have  $c \approx \boxed{1.96}$ .

**Problem 6.** The peak temperature  $T$ , as measured in degrees Fahrenheit, on a July day in New Jersey is a  $\mathcal{N}(85, 100)$  random variable.

Remark: Do not forget that, for our class, the second parameter in  $\mathcal{N}(\cdot, \cdot)$  is the variance (not the standard deviation).

- (a) Express the cdf of  $T$  in terms of the  $\Phi$  function.
- (b) Express each of the following probabilities in terms of the  $\Phi$  function(s). Make sure that the arguments of the  $\Phi$  functions are positive. (Positivity is required so that we can directly use the  $\Phi/Q$  tables to evaluate the probabilities.)
- (i)  $P[T > 100]$
- (ii)  $P[T < 60]$
- (iii)  $P[70 \leq T \leq 100]$
- (c) Express each of the probabilities in part (b) in terms of the  $Q$  function(s). Again, make sure that the arguments of the  $Q$  functions are positive.
- (d) Evaluate each of the probabilities in part (b) using the  $\Phi/Q$  tables.
- (e) Observe that the  $\Phi$  table (“Table 4” from the lecture) stops at  $z = 2.99$  and the  $Q$  table (“Table 5” from the lecture) starts at  $z = 3.00$ . Why is it better to give a table for  $Q(z)$  instead of  $\Phi(z)$  when  $z$  is large?

**Solution:**

- (a) Recall that when  $X \sim \mathcal{N}(m, \sigma^2)$ ,  $F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$ . Here,  $T \sim \mathcal{N}(85, 10^2)$ . Therefore,

$$F_T(t) = \Phi\left(\frac{t-85}{10}\right).$$

(b)

- (i)  $P[T > 100] = 1 - P[T \leq 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) = 1 - \Phi(1.5)$

(ii)  $P[T < 60] = P[T \leq 60]$  because  $T$  is a continuous random variable and hence  $P[T = 60] = 0$ . Now,  $P[T \leq 60] = F_T(60) = \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) = \boxed{1 - \Phi(2.5)}$ . Note that, for the last equality, we use the fact that  $\Phi(-z) = 1 - \Phi(z)$ .

(iii)

$$\begin{aligned} P[70 \leq T \leq 100] &= F_T(100) - F_T(70) = \Phi\left(\frac{100-85}{10}\right) - \Phi\left(\frac{70-85}{10}\right) \\ &= \Phi(1.5) - \Phi(-1.5) = \Phi(1.5) - (1 - \Phi(1.5)) = \boxed{2\Phi(1.5) - 1}. \end{aligned}$$

(c) In this question, we use the fact that  $Q(x) = 1 - \Phi(x)$ .

(i)  $1 - \Phi(1.5) = \boxed{Q(1.5)}$ .

(ii)  $1 - \Phi(2.5) = \boxed{Q(2.5)}$ .

(iii)  $2\Phi(1.5) - 1 = 2(1 - Q(1.5)) - 1 = 2 - 2Q(1.5) - 1 = \boxed{1 - 2Q(1.5)}$ .

(d)

(i)  $1 - \Phi(1.5) = 1 - 0.9332 = \boxed{0.0668}$ .

(ii)  $1 - \Phi(2.5) = 1 - 0.99379 = \boxed{0.0062}$ .

(iii)  $2\Phi(1.5) - 1 = 2(0.9332) - 1 = \boxed{0.8664}$ .

(e) When  $z$  is large,  $\Phi(z)$  will start with 0.999... The first few significant digits will all be the same and hence not quite useful to be there.

**Problem 7.** Suppose that the time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with  $\lambda = 0.0003$ .

(a) What proportion of the fans will last at least 10,000 hours?

(b) What proportion of the fans will last at most 7000 hours?

[Montgomery and Runger, 2010, Q4-97]

**Solution:** Let  $T$  be the time to failure (in hours). We are given that  $T \sim \mathcal{E}(\lambda)$  where  $\lambda = 3 \times 10^{-4}$ . Therefore,

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Here, we want to find  $P[T > 10^4]$ .

We shall first provide the general formula for the cdf  $P[T > t]$  when  $t > 0$ :

$$P[T > t] = \int_t^{\infty} f_T(\tau) d\tau = \int_t^{\infty} \lambda e^{-\lambda\tau} d\tau = -e^{-\lambda\tau} \Big|_t^{\infty} = e^{-\lambda t}. \quad (12.1)$$

Therefore,

$$P[T > 10^4] = e^{-3 \times 10^{-4} \times 10^4} = \boxed{e^{-3} \approx 0.0498}.$$

(b) We start with  $P[T \leq 7000] = 1 - P[T > 7000]$ . Next, we apply (12.1) to get

$$P[T \leq 7000] = 1 - P[T > 7000] = 1 - e^{-3 \times 10^{-4} \times 7000} = \boxed{1 - e^{-2.1} \approx 0.8775}.$$

## HW Solution 13 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

**Problem 1.** The input  $X$  and output  $Y$  of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$x \backslash y$	2	4	5
1	0.02	0.10	0.08
3	0.08	0.32	0.40

(a) Evaluate the following quantities:

- (i) The marginal pmf  $p_X(x)$
- (ii) The marginal pmf  $p_Y(y)$
- (iii)  $\mathbb{E}X$
- (iv)  $\text{Var } X$
- (v)  $\mathbb{E}Y$
- (vi)  $\text{Var } Y$
- (vii)  $P[XY < 6]$
- (viii)  $P[X = Y]$
- (ix)  $\mathbb{E}[XY]$
- (x)  $\mathbb{E}[(X - 3)(Y - 2)]$
- (xi)  $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$
- (xii)  $\text{Cov}[X, Y]$
- (xiii)  $\rho_{X,Y}$

(b) Find  $\rho_{X,X}$

(c) Calculate the following quantities using the values of  $\text{Var } X$ ,  $\text{Cov}[X, Y]$ , and  $\rho_{X,Y}$  that you got earlier.

- (i)  $\text{Cov}[3X + 4, 6Y - 7]$

- (ii)  $\rho_{3X+4,6Y-7}$
- (iii)  $\text{Cov}[X, 6X - 7]$
- (iv)  $\rho_{X,6X-7}$

**Solution:**

(a) The MATLAB codes are provided in the file P\_XY\_EVarCov.m.

(i) The marginal pmf  $p_X(x)$  is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

(ii) The marginal pmf  $p_Y(y)$  is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(iii)  $\mathbb{E}X = \sum_x xp_X(x) = 1 \times 0.2 + 3 \times 0.8 = 0.2 + 2.4 = \boxed{2.6}$ .

(iv)  $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = 1^2 \times 0.2 + 3^2 \times 0.8 = 0.2 + 7.2 = 7.4$ .

So,  $\text{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 7.4 - (2.6)^2 = 7.4 - 6.76 = \boxed{0.64}$ .

(v)  $\mathbb{E}Y = \sum_y yp_Y(y) = 2 \times 0.1 + 4 \times 0.42 + 5 \times 0.48 = 0.2 + 1.68 + 2.4 = \boxed{4.28}$ .

(vi)  $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = 2^2 \times 0.1 + 4^2 \times 0.42 + 5^2 \times 0.48 = 19.12$ .

So,  $\text{Var} Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 19.12 - 4.28^2 = \boxed{0.8016}$ .

(vii) Among the 6 possible pairs of  $(x, y)$  shown in the joint pmf matrix, only the pairs  $(1, 2)$ ,  $(1, 4)$ ,  $(1, 5)$  satisfy  $xy < 6$ . Therefore,  $[XY < 6] = [X = 1]$  which implies  $P[XY < 6] = P[X = 1] = \boxed{0.2}$ .

(viii) Among the 6 possible pairs of  $(x, y)$  shown in the joint pmf matrix, there is no pair which has  $x = y$ . Therefore,  $P[X = Y] = \boxed{0}$ .

(ix) First, we calculate the values of  $x \times y$ :

$x \setminus y$	2	4	5
1	2	4	5
3	6	12	15

Then, each  $x \times y$  is weighted (multiplied) by the corresponding probability  $p_{X,Y}(x, y)$ :

$$\begin{array}{c|ccc} x \setminus y & 2 & 4 & 5 \\ \hline 1 & 0.04 & 0.40 & 0.40 \\ 3 & 0.48 & 3.84 & 6.00 \end{array}$$

Finally,  $\mathbb{E}[XY]$  is sum of these numbers. Therefore,  $\mathbb{E}[XY] = \boxed{11.16}$ .

(x) First, we calculate the values of  $(x - 3) \times (y - 2)$ :

$$\begin{array}{c|ccc} x \setminus y & 2 & 4 & 5 \\ \hline 1 & 0 & -4 & -6 \\ 3 & 0 & 0 & 0 \end{array}$$

Then, each  $(x - 3) \times (y - 2)$  is weighted (multiplied) by the corresponding probability  $p_{X,Y}(x, y)$ :

$$\begin{array}{cc|ccc} & y - 2 & 0 & 2 & 3 \\ x - 3 & x \setminus y & 2 & 4 & 5 \\ \hline -2 & 1 & 0 & -0.40 & -0.48 \\ 0 & 3 & 0 & 0 & 0 \end{array}$$

Finally,  $\mathbb{E}[(X - 3)(Y - 2)]$  is sum of these numbers. Therefore,

$$\mathbb{E}[(X - 3)(Y - 2)] = \boxed{-0.88}$$

(xi) First, we calculate the values of  $x(y^3 - 11y^2 + 38y)$ :

$$\begin{array}{c|ccc} y^3 - 11y^2 + 38y & 40 & 40 & 40 \\ x \setminus y & 2 & 4 & 5 \\ \hline 1 & 40 & 40 & 40 \\ 3 & 120 & 120 & 120 \end{array}$$

Then, each  $x(y^3 - 11y^2 + 38y)$  is weighted (multiplied) by the corresponding probability  $p_{X,Y}(x, y)$ :

$$\begin{array}{c|ccc} x \setminus y & 2 & 4 & 5 \\ \hline 1 & 0.8 & 4.0 & 3.2 \\ 3 & 9.6 & 38.4 & 48.0 \end{array}$$

Finally,  $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$  is sum of these numbers. Therefore,

$$\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)] = \boxed{104}$$



$$(xii) \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y = 11.16 - (2.6)(4.28) = \boxed{0.032}.$$

$$(xiii) \rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sigma_X\sigma_Y} = \frac{0.032}{\sqrt{0.64}\sqrt{0.8016}} = \boxed{0.044677}$$

$$(b) \rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X\sigma_X} = \frac{\text{Var}[X]}{\sigma_X^2} = \boxed{1}.$$

(c)

$$(i) \text{Cov}[3X + 4, 6Y - 7] = 3 \times 6 \times \text{Cov}[X, Y] \approx 3 \times 6 \times 0.032 \approx \boxed{0.576}.$$

(ii) Note that

$$\begin{aligned} \rho_{aX+b, cY+d} &= \frac{\text{Cov}[aX + b, cY + d]}{\sigma_{aX+b}\sigma_{cY+d}} \\ &= \frac{ac\text{Cov}[X, Y]}{|a|\sigma_X|c|\sigma_Y} = \frac{ac}{|ac|}\rho_{X,Y} = \text{sign}(ac) \times \rho_{X,Y}. \end{aligned}$$

$$\text{Hence, } \rho_{3X+4, 6Y-7} = \text{sign}(3 \times 4)\rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}.$$

$$(iii) \text{Cov}[X, 6X - 7] = 1 \times 6 \times \text{Cov}[X, X] = 6 \times \text{Var}[X] \approx \boxed{3.84}.$$

$$(iv) \rho_{X, 6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}.$$

**Problem 2.** Suppose  $X \sim \text{binomial}(5, 1/3)$ ,  $Y \sim \text{binomial}(7, 4/5)$ , and  $X \perp\!\!\!\perp Y$ . Evaluate the following quantities.

$$(a) \mathbb{E}[(X - 3)(Y - 2)]$$

$$(b) \text{Cov}[X, Y]$$

$$(c) \rho_{X,Y}$$

**Solution:**

- (a) First, because  $X$  and  $Y$  are independent, we have  $\mathbb{E}[(X - 3)(Y - 2)] = \mathbb{E}[X - 3]\mathbb{E}[Y - 2]$ . Recall that  $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ . Therefore,  $\mathbb{E}[X - 3]\mathbb{E}[Y - 2] = (\mathbb{E}[X] - 3)(\mathbb{E}[Y] - 2)$ . Now, for Binomial( $n, p$ ), the expected value is  $np$ . So,

$$(\mathbb{E}[X] - 3)(\mathbb{E}[Y] - 2) = \left(5 \times \frac{1}{3} - 3\right) \left(7 \times \frac{4}{5} - 2\right) = -\frac{4}{3} \times \frac{18}{5} = \boxed{-\frac{24}{5}} = -4.8.$$

$$(b) \text{Cov}[X, Y] = \boxed{0} \text{ because } X \perp\!\!\!\perp Y.$$

$$(c) \rho_{X,Y} = \boxed{0} \text{ because } \text{Cov}[X, Y] = 0$$

## Extra Questions

Here are some optional questions for those who want more practice.

**Problem 3.** Let a continuous random variable  $X$  denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of  $X$  is

$$f_X(x) = \begin{cases} 5, & 4.9 \leq x \leq 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the probability that a current measurement is less than 5 milliamperes.
- Find and plot the cumulative distribution function of the random variable  $X$ .
- Find the expected value of  $X$ .
- Find the variance and the standard deviation of  $X$ .
- Find the expected value of power when the resistance is 100 ohms?

**Solution:** See handwritten solution.

**Problem 4.** Let  $X$  be a uniform random variable on the interval  $[0, 1]$ . Set

$$A = \left[0, \frac{1}{2}\right), \quad B = \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), \quad \text{and } C = \left[0, \frac{1}{8}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right) \cup \left[\frac{1}{2}, \frac{5}{8}\right) \cup \left[\frac{3}{4}, \frac{7}{8}\right).$$

Are the events  $[X \in A]$ ,  $[X \in B]$ , and  $[X \in C]$  independent?

**Solution:** Note that

$$\begin{aligned} P[X \in A] &= \int_0^{\frac{1}{2}} dx = \frac{1}{2}, \\ P[X \in B] &= \int_0^{\frac{1}{4}} dx + \int_{\frac{1}{2}}^{\frac{3}{4}} dx = \frac{1}{2}, \quad \text{and} \\ P[X \in C] &= \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx + \int_{\frac{3}{4}}^{\frac{7}{8}} dx = \frac{1}{2}. \end{aligned}$$

Now, for pairs of events, we have

$$P([X \in A] \cap [X \in B]) = \int_0^{\frac{1}{4}} dx = \frac{1}{4} = P[X \in A] \times P[X \in B], \quad (13.1)$$

$$P([X \in A] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{4}}^{\frac{3}{8}} dx = \frac{1}{4} = P[X \in A] \times P[X \in C], \text{ and} \quad (13.2)$$

$$P([X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx + \int_{\frac{1}{2}}^{\frac{5}{8}} dx = \frac{1}{4} = P[X \in B] \times P[X \in C]. \quad (13.3)$$

Finally,

$$P([X \in A] \cap [X \in B] \cap [X \in C]) = \int_0^{\frac{1}{8}} dx = \frac{1}{8} = P[X \in A] P[X \in B] P[X \in C]. \quad (13.4)$$

From (13.1), (13.2), (13.3) and (13.4), we can conclude that the events  $[X \in A]$ ,  $[X \in B]$ , and  $[X \in C]$  are independent.

**Problem 5.** Cholesterol is a fatty substance that is an important part of the outer lining (membrane) of cells in the body of animals. Its normal range for an adult is 120–240 mg/dl. The Food and Nutrition Institute of the Philippines found that the total cholesterol level for Filipino adults has a mean of 159.2 mg/dl and 84.1% of adults have a cholesterol level below 200 mg/dl. Suppose that the cholesterol level in the population is normally distributed.

- (a) Determine the standard deviation of this distribution.
- (b) What is the value of the cholesterol level that exceeds 90% of the population?
- (c) An adult is at moderate risk if cholesterol level is more than one but less than two standard deviations above the mean. What percentage of the population is at moderate risk according to this criterion?
- (d) An adult is thought to be at high risk if his cholesterol level is more than two standard deviations above the mean. What percentage of the population is at high risk?

**Solution:** Let  $X$  be the cholesterol level of a randomly chosen adult. It is given that  $X \sim \mathcal{N}(m, \sigma^2)$  where  $m = 159.2$  mg/dl. We also know that  $P[X < 200] = 0.841$ .

- (a) For any Gaussian random variable,  $P[X < 200] = \Phi\left(\frac{200-m}{\sigma}\right)$ . It is given that this probability should be 0.841. Our plan is then to first find the number  $z$  whose  $\Phi(z) = 0.841$ . Then, solve for  $\sigma$  from  $z = \frac{200-m}{\sigma}$ .

From the  $\Phi$  table,  $\Phi(0.99) \approx 0.8389$  and  $\Phi(1) \approx 0.8413$ . Because 0.841 is closer to 0.8413 than 0.8389, we conclude that the value of  $z$  that makes  $\Phi(z) = 0.841$  is  $z \approx 1$ .

From  $z = \frac{200-m}{\sigma}$ , plugging-in  $z \approx 1$  gives  $\sigma \approx 200 - m = 200 - 159.2 = \boxed{40.8 \text{ mg/dl}}$ .

- (b) Here, we want to find the value of  $x$  such that  $P[X \leq x] = 0.9$ . (90% of the population has cholesterol level lower than this  $x$ .)

For any Gaussian random variable,  $P[X \leq x] = \Phi\left(\frac{x-m}{\sigma}\right)$ . Our plan is then to first find the number  $z$  whose  $\Phi(z) = 0.9$ . Then, solve for  $x$  from  $z = \frac{x-m}{\sigma}$ .

From the  $\Phi$  table,  $\Phi(1.28) \approx 0.8997$  and  $\Phi(1.29) \approx 0.9015$ . Because 0.9 is closer to 0.8997 than 0.9015, we conclude that the value of  $z$  that makes  $\Phi(z) = 0.9$  is  $z \approx 1.28$ .

From  $z = \frac{x-m}{\sigma}$ , plugging-in  $z \approx 1.28$  gives  $x \approx 1.28\sigma + m \approx \boxed{211.424 \text{ mg/dl}}$ .

- (c) Here we want to find the probability that  $m + \sigma < X < m + 2\sigma$ :

$$P[m + \sigma < X < m + 2\sigma] = F_X(m + 2\sigma) - F_X(m + \sigma).$$

For any Gaussian random variable,  $F_X(x) = \Phi\left(\frac{x-m}{\sigma}\right)$ . Therefore,

$$\begin{aligned} P[m + \sigma < X < m + 2\sigma] &= \Phi\left(\frac{(m + 2\sigma) - m}{\sigma}\right) - \Phi\left(\frac{(m + \sigma) - m}{\sigma}\right) \\ &= \Phi(2) - \Phi(1) \approx 0.97725 - 0.8413 \approx \boxed{0.1359 = 13.59\%} \end{aligned}$$

- (d)  $P[X > m + 2\sigma] = 1 - P[X \leq m + 2\sigma] = 1 - F_X(m + 2\sigma) = 1 - \Phi\left(\frac{(m+2\sigma)-m}{\sigma}\right) = 1 - \Phi(2) \approx 1 - 0.97725 \approx \boxed{0.0228 = 2.28\%}$ .

**Problem 6** (Q3.5.6). Solve this question using the  $\Phi/Q$  table.

A professor pays 25 cents for each blackboard error made in lecture to the student who points out the error. In a career of  $n$  years filled with blackboard errors, the total amount in dollars paid can be approximated by a Gaussian random variable  $Y_n$  with expected value  $40n$  and variance  $100n$ .

- (a) What is the probability that  $Y_{20}$  exceeds 1000?  
 (b) How many years  $n$  must the professor teach in order that  $P[Y_n > 1000] > 0.99$ ?

**Solution:** We are given<sup>1</sup> that  $Y_n \sim \mathcal{N}(40n, 100n)$ . Recall that when  $X \sim \mathcal{N}(m, \sigma^2)$ ,

$$F_X(x) = \Phi\left(\frac{x - m}{\sigma}\right). \quad (13.5)$$

- (a) Here  $n = 20$ . So, we have  $Y_n \sim \mathcal{N}(40 \times 20, 100 \times 20) = \mathcal{N}(800, 2000)$ . For this random variable  $m = 800$  and  $\sigma = \sqrt{2000}$ .

We want to find  $P[Y_{20} > 1000]$  which is the same as  $1 - P[Y_{20} \leq 1000]$ . Expressing this quantity using cdf, we have

$$P[Y_{20} > 1000] = 1 - F_{Y_{20}}(1000).$$

Apply (13.5) to get

$$P[Y_{20} > 1000] = 1 - \Phi\left(\frac{1000 - 800}{\sqrt{2000}}\right) = 1 - \Phi(4.472) \approx Q(4.47) \approx \boxed{3.91 \times 10^{-6}}.$$

- (b) Here, the value of  $n$  is what we want. So, we will need to keep the formula in the general form. Again, from (13.5), for  $Y_n \sim \mathcal{N}(40n, 100n)$ , we have

$$P[Y_n > 1000] = 1 - F_{Y_n}(1000) = 1 - \Phi\left(\frac{1000 - 40n}{10\sqrt{n}}\right) = 1 - \Phi\left(\frac{100 - 4n}{\sqrt{n}}\right).$$

To find the value of  $n$  such that  $P[Y_n > 1000] > 0.99$ , we will first find the value of  $z$  which make

$$1 - \Phi(z) > 0.99. \quad (13.6)$$

At this point, we may try to solve for the value of  $Z$  by noting that (13.6) is the same as

$$\Phi(z) < 0.01. \quad (13.7)$$

Unfortunately, the tables that we have start with  $\Phi(0) = 0.5$  and increase to something close to 1 when the argument of the  $\Phi$  function is large. This means we can't directly find 0.01 in the table. Of course, 0.99 is in there and therefore we will need to solve (13.6) via another approach.

To do this, we use another property of the  $\Phi$  function. Recall that  $1 - \Phi(z) = \Phi(-z)$ . Therefore, (13.6) is the same as

$$\Phi(-z) > 0.99. \quad (13.8)$$

---

<sup>1</sup>Note that the expected value and the variance in this question are proportional to  $n$ . This naturally occurs when we consider the sum of i.i.d. random variables. The approximation by Gaussian random variable is a result of the central limit theorem (CLT).

From our table, we can then conclude that (13.7) (which is the same as (13.8)) will happen when  $-z > 2.33$ . (If you have MATLAB, then you can get a more accurate answer of 2.3263.)

Now, plugging in  $z = \frac{100-4n}{\sqrt{n}}$ , we have  $\frac{4n-100}{\sqrt{n}} > 2.33$ . To solve for  $n$ , we first let  $x = \sqrt{n}$ . In which case, we have  $\frac{4x^2-100}{x} > 2.33$  or, equivalently,  $4x^2 - 2.33x - 100 > 0$ . The two roots are  $x = -4.717$  and  $x > 5.3$ . So, We need  $x < -4.717$  or  $x > 5.3$ . Note that  $x = \sqrt{n}$  and therefore can not be negative. So, we only have one case; that is, we need  $x > 5.3$ . Because  $n = x^2$ , we then conclude that we need  $n > 28.1$  years.

**Problem 7.** The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Determine the probability density function of  $X$ .
- (b) What proportion of reactions is complete within 200 milliseconds?

**Solution:** Note that the cdf  $F_X(x)$  is a continuous function. Therefore,  $X$  is a continuous RV.

$$(a) f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 0.01e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

At  $x = 0$ , the derivative does not exist. However,  $X$  is a continuous RV. Therefore, we can assign  $f_X(0)$  to be any arbitrary value. Here, we set  $f_X(0) = 0$ :

$$f_X(x) = \begin{cases} 0.01e^{-0.01x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$(b) P[X < 200] = P[X \leq 200] = F_X(200) = 1 - e^{-0.01 \times 200} = 1 - e^{-2} \approx 0.8647.$$

**Problem 8.** Suppose  $X \sim \text{binomial}(5, 1/3)$ ,  $Y \sim \text{binomial}(7, 4/5)$ , and  $X \perp\!\!\!\perp Y$ .

- (a) A vector describing the pmf of  $X$  can be created by the MATLAB expression:

$$\mathbf{x} = 0:5; \mathbf{pX} = \text{binopdf}(\mathbf{x}, 5, 1/3).$$

What is the expression that would give  $\mathbf{pY}$ , a corresponding vector describing the pmf of  $Y$ ?

- (b) Use  $\mathbf{pX}$  and  $\mathbf{pY}$  from part (a), how can you create the joint pmf matrix in MATLAB? Do not use “for-loop”, “while-loop”, “if statement”. Hint: Multiply them in an appropriate orientation.
- (c) Use **MATLAB** to evaluate the following quantities. Again, do not use “for-loop”, “while-loop”, “if statement”.
- $\mathbb{E}X$
  - $P[X = Y]$
  - $P[XY < 6]$

**Solution:** The MATLAB codes are provided in the file `P_XY_jointfromMarginal_indp.m`.

- (a) `y = 0:7; pY = binopdf(y,7,4/5);`
- (b) `P = pX.'*pY;`
- (c)
- $\mathbb{E}X = 1.667$
  - $P[X = Y] = 0.0121$
  - $P[XY < 6] = 0.2727$

**Problem 9.** Suppose  $\text{Var } X = 5$ . Find  $\text{Cov}[X, X]$  and  $\rho_{X,X}$ .

**Solution:**

- (a)  $\text{Cov}[X, X] = \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)] = \mathbb{E}[(X - \mathbb{E}X)^2] = \text{Var } X = 5$ .
- (b)  $\rho_{X,X} = \frac{\text{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\text{Var } X}{\sigma_X^2} = \frac{\text{Var } X}{\text{Var } X} = 1$ .

**Problem 10.** Suppose we know that  $\sigma_X = \frac{\sqrt{21}}{10}$ ,  $\sigma_Y = \frac{4\sqrt{6}}{5}$ ,  $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$ .

- (a) Find  $\text{Var}[X + Y]$ .
- (b) Find  $\mathbb{E}[(Y - 3X + 5)^2]$ . Assume  $\mathbb{E}[Y - 3X + 5] = 1$ .

**Solution:**

- (a) First, we know that  $\text{Var } X = \sigma_X^2 = \frac{21}{100}$ ,  $\text{Var } Y = \sigma_Y^2 = \frac{96}{25}$ , and  $\text{Cov}[X, Y] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$ . Now,

$$\begin{aligned} \text{Var}[X + Y] &= \mathbb{E} [((X + Y) - \mathbb{E}[X + Y])^2] = \mathbb{E} [(X - \mathbb{E}X) + (Y - \mathbb{E}Y)]^2 \\ &= \mathbb{E} [(X - \mathbb{E}X)^2] + 2\mathbb{E} [(X - \mathbb{E}X)(Y - \mathbb{E}Y)] + \mathbb{E} [(Y - \mathbb{E}Y)^2] \\ &= \text{Var } X + 2\text{Cov}[X, Y] + \text{Var } Y \\ &= \boxed{\frac{389}{100}} = 3.89. \end{aligned}$$

Remark: It is useful to remember that

$$\text{Var}[X + Y] = \text{Var } X + 2\text{Cov}[X, Y] + \text{Var } Y.$$

Note that when  $X$  and  $Y$  are uncorrelated,  $\text{Var}[X + Y] = \text{Var } X + \text{Var } Y$ . This simpler formula also holds when  $X$  and  $Y$  are independence because independence is a stronger condition.

- (b) First, we write

$$Y - aX - b = (Y - \mathbb{E}Y) - a(X - \mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_c.$$

Now, using the expansion

$$(u + v + t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned} (Y - aX - b)^2 &= (Y - \mathbb{E}Y)^2 + a^2(X - \mathbb{E}X)^2 + c^2 \\ &\quad - 2a(X - \mathbb{E}X)(Y - \mathbb{E}Y) - 2c(Y - \mathbb{E}Y) + 2a(X - \mathbb{E}X)c. \end{aligned}$$

Recall that  $\mathbb{E}[X - \mathbb{E}X] = \mathbb{E}[Y - \mathbb{E}Y] = 0$ . Therefore,

$$\mathbb{E}[(Y - aX - b)^2] = \text{Var } Y + a^2 \text{Var } X + c^2 - 2a\text{Cov}[X, Y]$$

Plugging back the value of  $c$ , we have

$$\boxed{\mathbb{E}[(Y - aX - b)^2] = \text{Var } Y + a^2 \text{Var } X + (\mathbb{E}[(Y - aX - b)])^2 - 2a\text{Cov}[X, Y]}.$$

Here,  $a = 3$  and  $b = -5$ . Plugging these values along with the given quantities into the formula gives

$$\mathbb{E}[(Y - aX - b)^2] = \boxed{\frac{721}{100}} = 7.21.$$



**Problem 11.** The input  $X$  and output  $Y$  of a system subject to random perturbations are described probabilistically by the joint pmf  $p_{X,Y}(x, y)$ , where  $x = 1, 2, 3$  and  $y = 1, 2, 3, 4, 5$ . Let  $P$  denote the joint pmf matrix whose  $i, j$  entry is  $p_{X,Y}(i, j)$ , and suppose that

$$P = \frac{1}{71} \begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

- Find the marginal pmfs  $p_X(x)$  and  $p_Y(y)$ .
- Find  $\mathbb{E}X$
- Find  $\mathbb{E}Y$
- Find  $\text{Var } X$
- Find  $\text{Var } Y$

**Solution:** All of the calculations in this question are simply plugging numbers into appropriate formula. The MATLAB codes are provided in the file `P_XY_marginal_2.m`.

- The marginal pmf  $p_X(x)$  is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf  $p_Y(y)$  is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

- $\mathbb{E}X = \frac{136}{71} \approx 1.9155$
- $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$
- $\text{Var } X = \frac{3230}{5041} \approx 0.6407$
- $\text{Var } Y = \frac{9220}{5041} \approx 1.8290$