ECS 315: Probability and Random Processes **2016/1**

HW Solution 7 — Due: Oct 25, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. For each description of a random variable X below, indicate whether X is a discrete random variable.

(a) X is the number of websites visited by a randomly chosen software engineer in a day.

(b) X is the number of classes a randomly chosen student is taking.

(c) X is the average height of the passengers on a randomly chosen bus.

(d) A game involves a circular spinner with eight sections labeled with numbers. X is the amount of time the spinner spins before coming to a rest.

(e) X is the thickness of the longest book in a randomly chosen library.

(f) X is the number of keys on a randomly chosen keyboard.

(g) X is the length of a randomly chosen person's arm.

Solution:We consider the number of possibilities for the values of X in each part. The X defined in parts (a), (b), and (f) are discrete. The X defined in other parts are not discrete.

Problem 2. An optical inspection system is to distinguish among different part types. The probability of a correct classification of any part is 0.98. Suppose that three parts are inspected and that the classifications are independent.

(a) Let the random variable X denote the number of parts that are correctly classified. Determine the probability mass function of X. [Montgomery and Runger, 2010, Q3-20]

(b) Let the random variable Y denote the number of parts that are incorrectly classified. Determine the probability mass function of Y.

Solution[1]:

We will reexpress the problem in terms of Bernoulli trials so that we can use the results discussed in class. In this problem, we have three Bernoulli trials. Each trial deals with classification.

[1]The solution provided here assumes that we still haven't reached the part of the course where binomial random variable is discussed. Therefore, the pmf is derived by relying on the concept of Bernoulli trials and the formula discussed back when we studied that topic.

(a) To find $p_X(x)$, first we find its support. Three parts are inspected here. Therefore, X can be 0, 1, 2, or 3. So, we need to find $p_X(x) = P[X = x]$ when $x = 0, 1, 2$ or 3. The pmf $p_X(x)$ for other x values are all 0 because X cannot take the value of those x.

For each $x \in \{0, 1, 2, 3\}$, $p_X(x) = P[X = x]$ is simply the probability that exactly x parts are correctly classified. Note that, because we are interested in the correctly classified part, we define the "success" event for a trial to be the event that the part is classified correctly. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is $p = 0.98$. Under such interpretation (of "success"), $p_X(x)$ is then the same as finding the probability of having exactly x successes in $n = 3$ Bernoulli trials. We have seen in class that the probability of this is $\binom{3}{x}p^x(1-p)^{3-x}$. Plugging in $p = 0.98$, we have $p_X(x) = \binom{3}{x}0.98^x(0.02)^{3-x}$ for $x \in \{0, 1, 2, 3\}$.

Combining the expression above with the cases for other x values, we then have

$$p_X(x) = \begin{cases} \binom{3}{x}0.98^x(0.02)^{3-x}, & x \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{7.1}$$

In particular, $p_X(0) = 8 \times 10^{-6}$, $p_X(1) = 0.001176$, $p_X(2) = 0.057624$, and $p_X(3) = 0.941192$.

Remark: In fact, this X is a binomial random variable with $n = 3$ and $p = 0.98$. In `MATLAB`, the probabilities above can be calculated via the command `binopdf(0:3,3,0.98)`.

(b) ***Method 1***: Similar analysis is performed on the random variable Y. The only difference here is that, now, we are interested in the number of parts that are *incorrectly* classified. Therefore, we will define the "success" event for a trial to be the event that the part is classified *incorrectly*. We are given that the probability of a correct classification of any part is 0.98. Therefore, for each of our Bernoulli trials, the probability of success is $1 - p = 1 - 0.98 = 0.02$. With this new probability of success, we have

$$p_Y(y) = \begin{cases} \binom{3}{y}0.02^y(0.98)^{3-y}, & y \in \{0,1,2,3\}, \\ 0, & \text{otherwise} \end{cases} \tag{7.2}$$

In particular, $p_Y(0) = 0.941192$, $p_Y(1) = 0.057624$, $p_Y(2) = 0.001176$, and $p_Y(3) = 8 \times 10^{-6}$.

Remark: In fact, this Y is a binomial random variable with $n = 3$ and $p = 0.02$. In `MATLAB`, the probability values above can be calculated via the command `binopdf(0:3,3,0.02)`.

Method 2: Alternatively, note that there are three parts. If X of them are classified correctly, then the number of incorrectly classified parts is $n - X$, which is what we defined as Y. Therefore, $Y = 3 - X$. Hence, $p_Y(y) = P[Y = y] = P[3 - X = y] = P[X = 3 - y] = p_X(3 - y)$.

Problem 3. Consider the sample space $\Omega = \{-2, -1, 0, 1, 2, 3, 4\}$. Suppose that $P(A) = |A|/|\Omega|$ for any event $A \subset \Omega$. Define the random variable $X(\omega) = \omega^2$. Find the probability mass function of X.

Solution: The random variable maps the outcomes $\omega = -2, -1, 0, 1, 2, 3, 4$ to numbers $x = 4, 1, 0, 1, 4, 9, 16$, respectively. Therefore,

$$
\begin{aligned}
p_X(0) &= P(\{\omega : X(\omega) = 0\}) = P(\{0\}) = \frac{1}{7}, \\
p_X(1) &= P(\{\omega : X(\omega) = 1\}) = P(\{-1, 1\}) = \frac{2}{7}, \\
p_X(4) &= P(\{\omega : X(\omega) = 4\}) = P(\{-2, 2\}) = \frac{2}{7}, \\
p_X(9) &= P(\{\omega : X(\omega) = 9\}) = P(\{3\}) = \frac{1}{7}, \text{ and} \\
p_X(16) &= P(\{\omega : X(\omega) = 16\}) = P(\{4\}) = \frac{1}{7}.
\end{aligned}
$$

Combining the results above, we get the complete pmf:

$$
p_X(x) = \boxed{\begin{cases} \frac{1}{7}, & x = 0, 9, 16, \\ \frac{2}{7}, & x = 1, 4, \\ 0, & \text{otherwise.} \end{cases}}
$$