ECS 315: Probability and Random Processes **2016/1**

HW Solution 6 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 1. In an experiment, A, B, C, and D are events with probabilities $P(A \cup B) = \frac{5}{8}$, $P(A) = \frac{3}{8}$, $P(C \cap D) = \frac{1}{3}$, and $P(C) = \frac{1}{2}$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find

(i) $P(A \cap B)$

(ii) $P(B)$

(iii) $P(A \cap B^c)$

(iv) $P(A \cup B^c)$

(b) Are A and B independent?

(c) Find

(i) $P(D)$

(ii) $P(C \cap D^c)$

(iii) $P(C^c \cap D^c)$

(iv) $P(C|D)$

(v) $P(C \cup D)$

(vi) $P(C \cup D^c)$

(d) Are C and D^c independent?

Solution:

(a)

(i) Because $A \perp B$, we have $A \cap B = \emptyset$ and hence $P(A \cap B) = \boxed{0}$.

(ii) Recall that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Hence, $P(B) = P(A \cup B) - P(A) + P(A \cap B) = 5/8 - 3/8 + 0 = 2/8 = \boxed{1/4}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = P(A) = \boxed{3/8}$.

(iv) Start with $P(A \cup B^c) = 1 - P(A^c \cap B)$. Now, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) = 1/4$. Hence, $P(A \cup B^c) = 1 - 1/4 = \boxed{3/4}$.

(b) Events A and B are not independent because $P(A \cap B) \neq P(A)P(B)$.

(c)

(i) Because $C \perp\!\!\!\perp D$, we have $P(C \cap D) = P(C)P(D)$. Hence, $P(D) = \frac{P(C \cap D)}{P(C)} = \frac{1/3}{1/2} = \boxed{2/3}$.

(ii) **Method 1**: $P(C \cap D^c) = P(C) - P(C \cap D) = 1/2 - 1/3 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C \perp\!\!\!\perp D^c$. Hence, $P(C \cap D^c) = P(C)P(D^c) = \frac{1}{2}\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iii) **Method 1**: First, we find $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = 5/6$. Hence, $P(C^c \cap D^c) = 1 - P(C \cup D) = 1 - 5/6 = \boxed{1/6}$.
Method 2: Alternatively, because $C \perp\!\!\!\perp D$, we know that $C^c \perp\!\!\!\perp D^c$. Hence, $P(C^c \cap D^c) = P(C^c)P(D^c) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right) = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$.

(iv) Because $C \perp\!\!\!\perp D$, we have $P(C|D) = P(C) = \boxed{1/2}$.

(v) In part (iii), we already found $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 1/2 + 2/3 - 1/3 = \boxed{5/6}$.

(vi) **Method 1**: $P(C \cup D^c) = 1 - P(C^c \cap D) = 1 - P(C^c)P(D) = 1 - \frac{1}{2}\frac{2}{3} = \boxed{2/3}$. Note that we use the fact that $C^c \perp\!\!\!\perp D$ to get the second equality.
Method 2: Alternatively, $P(C \cup D^c) = P(C) + P(D^c) - P(C \cap D^C)$. From (i), we have $P(D) = 2/3$. Hence, $P(D^c) = 1 - 2/3 = 1/3$. From (ii), we have $P(C \cap D^C) = 1/6$. Therefore, $P(C \cup D^c) = 1/2 + 1/3 - 1/6 = 2/3$.

(d) Yes. We know that if $C \perp\!\!\!\perp D$, then $C \perp\!\!\!\perp D^c$.

Problem 2. Series Circuit: The circuit in Figure 6.1 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-32]

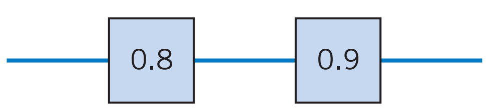

Figure 6.1: Circuit for Problem 2

Solution: Let L and R denote the events that the left and right devices operate, respectively. For a path to exist, both need to operate. Therefore, the probability that the circuit operates is $P(L \cap R)$.

We are told that $L^c \perp\!\!\!\perp R^c$. This is equivalent to $L \perp\!\!\!\perp R$. By their independence,

$$P(L \cap R) = P(L)P(R) = 0.8 \times 0.9 = \boxed{0.72}.$$

Problem 3 (Majority Voting in Digital Communication)**.** A certain binary communication system has a bit-error rate of 0.1; i.e., in transmitting a single bit, the probability of receiving the bit in error is 0.1. To transmit messages, a three-bit repetition code is used. In other words, to send the message 1, a "codeword" 111 is transmitted, and to send the message 0, a "codeword" 000 is transmitted. At the receiver, if two or more 1s are received, the decoder decides that message 1 was sent; otherwise, i.e., if two or more zeros are received, it decides that message 0 was sent.

Assuming bit errors occur independently, find the probability that the decoder puts out the wrong message.

[Gubner, 2006, Q2.62]

Solution: Let $p = 0.1$ be the bit error rate. Let $\mathcal{E}$ be the error event. (This is the event that the decoded bit value is not the same as the transmitted bit value.) Because majority voting is used, event $\mathcal{E}$ occurs if and only if there are at least two bit errors. Therefore

$$P(\mathcal{E}) = \binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 = p^2(3-2p).$$

When $p = 0.1$, we have $P(\mathcal{E}) \approx \boxed{0.028}$.

Problem 4. Consider a random experiment in which you roll a 20-sided fair dice. We define the following random variables from the outcomes of this experiment:

$$X(\omega) = \omega, \quad Y(\omega) = (\omega - 5)^2, \quad Z(\omega) = |\omega - 5| - 3$$

Evaluate the following probabilities:

(a) $P[X = 5]$

(b) $P[Y = 16]$

(c) $P[Y > 10]$

(d) $P[Z > 10]$

(e) $P[5 < Z < 10]$

Solution: In this question, $\Omega = \{1, 2, 3, \dots, 20\}$ because the dice has 20 sides. All twenty outcomes are equally-likely because the dice is fair. So, the probability of each outcome is $\frac{1}{20}$.

(a) From $X(\omega) = \omega$, we have $X(\omega) = 5$ if and only if $\omega = 5$. Therefore, $P[X = 5] = P[\{5\}] = \boxed{\frac{1}{20}}$.

(b) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) = 16$ if and only if $\omega = \pm 4 + 5 = 1$ or 9. Therefore, $P[Y = 16] = P[\{1, 9\}] = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(c) From $Y(\omega) = (\omega - 5)^2$, we have $Y(\omega) > 10$ if and only if $(\omega - 5)^2 > 10$. The values of ω that satisfy this condition are $1, 9, 10, 11, \dots, 20$. Therefore, $P[Y > 10] = P[\{1, 9, 10, 11, \dots, 20\}] = \boxed{\frac{13}{20}}$.

(d) The values of ω that satisfy $|\omega - 5| - 3 > 10$ are 19 and 20. Therefore, $P[Z > 10] = P[\{19, 20\}] = \frac{2}{20} = \boxed{\frac{1}{10}}$.

(e) The values of ω that satisfy $5 < |\omega - 5| - 3 < 10$ are $14, 15, 16, 17$. Therefore, $P[5 < Z < 10] = P[\{14, 15, 16, 17\}] = \frac{4}{20} = \boxed{\frac{1}{5}}$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 5. In this question, each experiment has equiprobable outcomes.

(a) Let $\Omega = \{1, 2, 3, 4\}$, $A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{2, 3\}$.

- (i) Determine whether $P(A_i \cap A_j) = P(A_i) P(A_j)$ for all $i \neq j$.
- (ii) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$.
- (iii) Are A_1, A_2, and A_3 independent?

(b) Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2, 3, 4\}$, $A_2 = A_3 = \{4, 5, 6\}$.

- (i) Check whether $P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2) P(A_3)$.
- (ii) Check whether $P(A_i \cap A_j) = P(A_i) P(A_j)$ for all $i \neq j$.
- (iii) Are A_1, A_2, and A_3 independent?

Solution:

(a) We have $P(A_i) = \frac{1}{2}$ and $P(A_i \cap A_j) = \frac{1}{4}$.

  (i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for any $i \neq j$.

  (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$. Hence, $P(A_1 \cap A_2 \cap A_3) = 0$, which is *not* the same as $P(A_1)P(A_2)P(A_3)$.

  (iii) No. Although the three conditions for pairwise independence are satisfied, the last (product) condition for independence among three events is not.

Remark: This counter-example shows that pairwise independence does not imply independence.

(b) We have $P(A_1) = \frac{4}{6} = \frac{2}{3}$ and $P(A_2) = P(A_3) = \frac{3}{6} = \frac{1}{2}$.

  (i) $A_1 \cap A_2 \cap A_3 = \{4\}$. Hence, $P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}$.
  $P(A_1)P(A_2)P(A_3) = \frac{2}{3}\frac{1}{2}\frac{1}{2} = \frac{1}{6}$.
  Hence, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

  (ii) $P(A_2 \cap A_3) = P(A_2) = \frac{1}{2} \neq P(A_2)P(A_3)$
  $P(A_1 \cap A_2) = p(4) = \frac{1}{6} \neq P(A_1)P(A_2)$
  $P(A_1 \cap A_3) = p(4) = \frac{1}{6} \neq P(A_1)P(A_3)$
  Hence, $P(A_i \cap A_j) \neq P(A_i)P(A_j)$ for all $i \neq j$.

  (iii) No. TO be independent, the three events must satisfy four conditions. Here, only one is satisfied.

Remark: This counter-example shows that one product condition does not imply independence.

Problem 6. A Web ad can be designed from four different colors, three font types, five font sizes, three images, and five text phrases. A specific design is randomly generated by the Web server when you visit the site. Let A denote the event that the design color is red and let B denote the event that the font size is not the smallest one.

(a) Use classical probability to evaluate $P(A)$, $P(B)$ and $P(A \cap B)$. Show that the two events A and B are independent by checking whether $P(A \cap B) = P(A)P(B)$.

(b) Using the values of $P(A)$ and $P(B)$ from the previous part and the fact that $A \perp\!\!\!\perp B$, calculate the following probabilities.

  (i) $P(A \cup B)$

  (ii) $P(A \cup B^c)$

  (iii) $P(A^c \cup B^c)$

[Montgomery and Runger, 2010, Q2-84]
Solution:

(a) By multiplication rule, there are

$$|\Omega| = 4 \times 3 \times 5 \times 3 \times 5 \tag{6.1}$$

possible designs. The number of designs whose color is red is given by

$$|A| = 1 \times 3 \times 5 \times 3 \times 5.$$

Note that the "4" in (6.1) is replaced by "1" because we only consider one color (red). Therefore,

$$P(A) = \frac{1 \times 3 \times 5 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{1}{4}.}$$

Similarly, $|B| = 4 \times 3 \times 4 \times 3 \times 5$ where the "5" in the middle of (6.1) is replaced by "4" because we can't use the smallest font size. Therefore,

$$P(B) = \frac{4 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \boxed{\frac{4}{5}.}$$

For the event $A \cap B$, we replace "4" in (6.1) by "1" because we need red color and we replace "5" in the middle of (6.1) by "4" because we can't use the smallest font size. This gives

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1 \times 3 \times 4 \times 3 \times 5}{4 \times 3 \times 5 \times 3 \times 5} = \frac{1 \times 4}{4 \times 5} = \boxed{\frac{1}{5}} = 0.2.$$

Because $P(A \cap B) = P(A)P(B)$, the events A and B are independent.

(b)

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{4}{5} - \frac{1}{5} = \boxed{\frac{17}{20} = 0.85.}$

(ii) **Method 1**: $P(A \cup B^c) = 1 - P\left((A \cup B^c)^c\right) = 1 - P(A^c \cap B)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$. Hence, $P(A^c \cup B^c) = 1 - P(A^c)P(B) = 1 - \frac{3}{4}\frac{4}{5} = \frac{2}{5} = \boxed{0.4.}$

Method 2: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = B^c \cup (A \cap B)$. Therefore,

$$P(A \cup B^c) = P(B^c) + P(A \cap B) = 1 - P(B) + P(A)P(B) = 1 - \frac{4}{5} + \frac{1}{4}\frac{4}{5} = \frac{2}{5}.$$

Method 3: From the Venn diagram, note that $A \cup B^c$ can be expressed as a disjoint union: $A \cup B^c = A \cup (A^c \cap B^c)$. Therefore, $P(A \cup B^c) = P(A) + P(A^c \cap B^c)$. Because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B^c$. Hence,

$$P(A \cup B^c) = P(A) + P(A^c)P(B^c) = P(A) + (1 - P(A))(1 - P(B)) = \frac{1}{4} + \frac{3}{4}\frac{1}{5} = \frac{2}{5}.$$

(iii) **Method 1**: $P(A^c \cup B^c) = 1 - P((A^c \cup B^c)^c) = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$

Method 2: From the Venn diagram, note that $A^c \cup B^c$ can be expressed as a disjoint union: $A^c \cup B^c = (A^c \cap B) \cup (A \cap B^c) \cup (A^c \cap B^c)$. Therefore,

$$P(A^c \cup B^c) = P(A^c \cap B) + P(A \cap B^c) + P(A^c \cap B^c).$$

Now, because $A \perp\!\!\!\perp B$, we also have $A^c \perp\!\!\!\perp B$, $A \perp\!\!\!\perp B^c$, and $A^c \perp\!\!\!\perp B^c$. Hence,

$$\begin{aligned} P(A^c \cup B^c) &= P(A^c)P(B) + P(A)P(B^c) + P(A^c)P(B^c) \\ &= (1 - P(A))P(B) + P(A)(1 - P(B)) + (1 - P(A))(1 - P(B)) \\ &= \frac{3}{4} \times \frac{4}{5} + \frac{1}{4} \times \frac{1}{5} + \frac{3}{4} \times \frac{1}{5} = \frac{16}{20} = \frac{4}{5} \end{aligned}$$

Problem 7. Show that if A and B are independent events, then so are A and B^c, A^c and B, and A^c and B^c.

Solution: To show that two events C_1 and C_2 are independent, we need to show that $P(C_1 \cap C_2) = P(C_1)P(C_2)$.

(a) Note that

$$P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B).$$

Because $A \perp\!\!\!\perp B$, the last term can be factored in to $P(A)P(B)$ and hence

$$P(A \cap B^c) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)P(B^c)$$

(b) By interchanging the role of A and B in the previous part, we have

$$P(A^c \cap B) = P(B \cap A^c) = P(B)P(A^c).$$

(c) From set theory, we know that $A^c \cap B^c = (A \cup B)^c$. Therefore,

$$P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B),$$

where, for the last equality, we use

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

which is discussed in class.

Because $A \perp\!\!\!\perp B$, we have

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) \\ &= P(A^c)P(B^c). \end{aligned}$$

Remark: By interchanging the roles of A and A^c and/or B and B^c, it follows that if any one of the four pairs is independent, then so are the other three. [Gubner, 2006, p.31]

Problem 8. Anne and Betty go fishing. Find the conditional probability that Anne catches no fish given that at least one of them catches no fish. Assume they catch fish independently and that each has probability $0 < p < 1$ of catching no fish. [Gubner, 2006, Q2.62]

Hint: Let A be the event that Anne catches no fish and B be the event that Betty catches no fish. Observe that the question asks you to evaluate $P(A|(A \cup B))$.

Solution: From the question, we know that A and B are independent. The event "at least one of the two women catches nothing" can be represented by $A \cup B$. So we have

$$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A) + P(B) - P(A)P(B)} = \frac{p}{2p - p^2} = \boxed{\frac{1}{2-p}}.$$

Problem 9. The circuit in Figure 6.2 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-34]

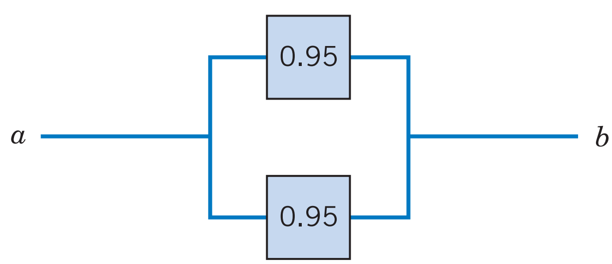

Figure 6.2: Circuit for Problem 9

Solution: Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. Therefore, the probability that the circuit operates is $P(T \cup B)$. Note that

$$P(T \cup B) = 1 - P((T \cup B)^c) = 1 - P(T^c \cap B^c).$$

We are told that $T^c \perp\!\!\!\perp B^c$. By their independence,

$$P(T^c \cap B^c) = P(T^c)P(B^c) = (1 - 0.95) \times (1 - 0.95) = 0.05^2 = 0.0025.$$

Therefore,

$$P(T \cup B) = 1 - P(T^c \cap B^c) = 1 - 0.0025 = \boxed{0.9975}.$$

Problem 10. The circuit in Figure 6.3 operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates? [Montgomery and Runger, 2010, Ex. 2-35]

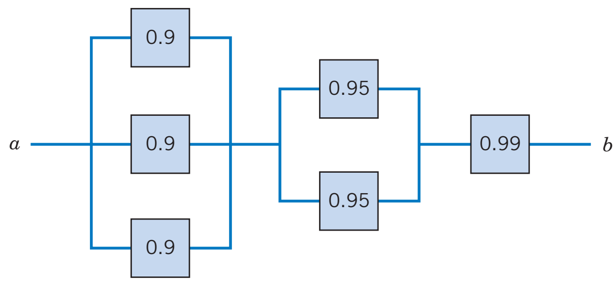

Figure 6.3: Circuit for Problem 10

Solution: The solution can be obtained from a partition of the graph into three columns. Let L denote the event that there is a path of functional devices only through the three units on the left. From the independence and based upon Problem 9,

$$P(L) = 1 - (1 - 0.9)^3 = 1 - 0.1^3 = 0.999.$$

Similarly, let M denote the event that there is a path of functional devices only through the two units in the middle. Then,

$$P(M) = 1 - (1 - 0.95)^2 = 1 - 0.05^2 = 1 - 0.0025 = 0.9975.$$

Finally, the probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99.

Therefore, with the independence assumption used again, along with similar reasoning to the solution of Problem 2, the solution is

$$0.999 \times 0.9975 \times 0.99 = 0.986537475 \approx \boxed{0.987}.$$