ECS 315: Probability and Random Processes **2016/1**

HW Solution 4 — Due: Sep 20, 5 PM

Lecturer: Prapun Suksompong, Ph.D.

Instructions

(a) This assignment has 6 pages.

(b) (1 pt) Write your first name and the last three digit of your student ID on the upper-right corner of ***every*** submitted page.

(c) (1 pt) For each part, write your explanation/derivation and answer in the space provided.

(d) (8 pt) It is important that you try to solve all non-optional problems.

(e) Late submission will be heavily penalized.

Problem 1. Let A and B be events for which $P(A)$, $P(B)$, and $P(A \cup B)$ are known. Express the following probabilities in terms of the three known probabilities above.

(a) $P(A \cap B)$

(b) $P(A \cap B^c)$

(c) $P(B \cup (A \cap B^c))$

(d) $P(A^c \cap B^c)$

Solution:

(a) $P(A \cap B) = \boxed{P(A) + P(B) - P(A \cup B)}$. This property is shown in class.

(b) We have seen[1] in class that $P(A \cap B^c) = P(A) - P(A \cap B)$. Plugging in the expression for $P(A \cap B)$ from the previous part, we have

$$P(A \cap B^c) = P(A) - (P(A) + P(B) - P(A \cup B)) = \boxed{P(A \cup B) - P(B)}.$$

[1]This shows up when we try to derive the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The key idea is that the set A can be expressed as a disjoint union between $A \cap B$ and $A \cap B^c$. Therefore, by finite additivity, $P(A) = P(A \cap B) + P(A \cap B^c)$. It is easier to visualize this via the Venn diagram.

Alternatively, we can start from scratch with the set identity $A \cup B = B \cup (A \cap B^c)$ whose union is a disjoint union. Hence,

$$P(A \cup B) = P(B) + P(A \cap B^c).$$

Moving $P(B)$ to the LHS finishes the proof.

(c) $P(B \cup (A \cap B^c)) = \boxed{P(A \cup B)}$ because $A \cup B = B \cup (A \cap B^c)$.

(d) $P(A^c \cap B^c) = \boxed{1 - P(A \cup B)}$ because $A^c \cap B^c = (A \cup B)^c$.

Problem 2. Continue from Problem 3 in HW3.

Recall that, there, we consider a random experiment whose sample space is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Find the following probabilities.

(a) $P(A|B)$

(b) $P(B|A)$

(c) $P(B|A^c)$

Solution: In HW2, we have already found

$$\begin{aligned} P(A) &= P(\{a,b,c\}) = 0.1 + 0.1 + 0.2 = 0.4, \\ P(B) &= P(\{c,d,e\}) = 0.2 + 0.4 + 0.2 = 0.8, \text{ and} \\ P(A \cap B) &= P(\{c\}) = 0.2. \end{aligned}$$

Therefore, by definition,

(a) $P(A|B) \equiv \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.8} = \boxed{\frac{1}{4}}$ and

(b) $P(B|A) \equiv \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.4} = \boxed{\frac{1}{2}}$.

(c) DO <u>*NOT*</u> start with $P(B|A^c) = 1 - P(B|A)$. This is not one of the formulas for conditional probabilities. Here, we will have to go back to the definition:

$$P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)} = \frac{P(\{d,e\})}{P(\{d,e\})} = \boxed{1}.$$

Problem 3.

(a) Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$ Determine the following:

(i) $P(A \cap B)$

(ii) $P(A^c \cap B)$

[Montgomery and Runger, 2010, Q2-105]

(b) Suppose that $P(A|B) = 0.2$, $P(A|B^c) = 0.3$ and $P(B) = 0.8$ What is $P(A)$? [Montgomery and Runger, 2010, Q2-106]

Solution:

(a)

(i) By definition, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.4 \times 0.5 = \boxed{0.2.}$$

(ii) $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = 0.5 - 0.2 = \boxed{0.3.}$
Alternatively, one can apply the property $P(A^c|B) = 1 - P(A|B)$ to get

$$P(A^c \cap B) = P(A^c|B)P(B) = (1 - P(A|B))P(B) = (1 - 0.4) \times 0.5 = 0.3.$$

(b) **Method 1**: By definition, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Therefore,

$$P(A \cap B) = P(A|B)P(B) = 0.2 \times 0.8 = 0.16.$$

Next, from $P(B) = 0.8$, we know that

$$P(B^c) = 1 - P(B) = 1 - 0.8 = 0.2.$$

By definition, $P(A|B^c) = \frac{P(A \cap B^c)}{P(B^c)}$. Therefore,

$$P(A \cap B^c) = P(A|B^c)P(B^c) = 0.3 \times 0.2 = 0.06.$$

Hence, $P(A) = P(A \cap B) + P(A \cap B^c) = 0.16 + 0.16 = \boxed{0.22.}$

Method 2: By the total probability formula, $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.2 \times 0.8 + 0.3 \times (1 - 0.8) = \boxed{0.22.}$.

Method 3: For those who are not seeking a "smart" way to solve this question, we can try the following:

Note that when we have two events, the sample space is always partitioned into four events: $A \cap B$, $A^c \cap B$, $A \cap B^c$, and $A^c \cap B^c$. (It might be helpful to draw the Venn diagram here.) Let's define their probabilities as p_1, p_2, p_3, and p_4, respectively. We are given three conditions which can then be turned into three equations. There is also one extra condition that $p_1 + p_2 + p_3 + p_4 = 1$. Therefore, we have four equations with four unknowns. Applying some high-school algebra, we should be able to solve for p_1, p_2, p_3, and p_4. With these, we can calculate probability of any event. For example, $P(A) = p_1 + p_3$.

Problem 4. Someone has rolled a fair dice twice. Suppose he tells you that "one of the rolls turned up a face value of six". What is the probability that the other roll turned up a six as well? [Tijms, 2007, Example 8.1, p. 244]

Hint: Note the followings:

- The answer is not $\frac{1}{6}$.
- Although there is no use of the word "give" or "conditioned on" in this question, the probability we seek is a conditional one. We have an extra piece of information because we know that the event "one of the rolls turned up a face value of six" has occurred.
- The question says "one of the rolls" without telling us which roll (the first or the second) it is referring to.

Solution: Let the sample space be the set $\{(i,j)|i,j = 1,\dots,6\}$, where i and j denote the outcomes of the first and second rolls, respectively. They are all equally likely; so each has probability of 1/36. The event of two sixes is given by $A = \{(6,6)\}$ and the event of at least one six is given by $B = (1,6),\dots,(5,6),(6,6),(6,5),\dots,(6,1)$. Applying the definition of conditional probability gives

$$P(A|B) = P(A\cap B)/P(B) = \frac{1/36}{11/36}.$$

Hence the desired probability is $\boxed{1/11}$.

Extra Question

Here is an optional question for those who want more practice.

Problem 5.

(a) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$. Find the range of possible values for $P(A\cap B)$. Hint: Smaller than the interval $[0,1]$. [Capinski and Zastawniak, 2003, Q4.21]

(b) Suppose that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Find the range of possible values for $P(A\cup B)$. Hint: Smaller than the interval $[0,1]$. [Capinski and Zastawniak, 2003, Q4.22]

Solution:

(a) We will try to derive general bounds for $P(A \cap B)$.

First, recall[2], from the lecture notes, that "$P(A \cap B)$ can not exceed $P(A)$ and $P(B)$":

$$P(A \cap B) \le \min\{P(A), P(B)\}. \tag{4.1}$$

On the other hand, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{4.2}$$

Now, $P(A \cup B)$ is a probability and hence its value must be between 0 and 1:

$$0 \le P(A \cup B) \le 1 \tag{4.3}$$

Combining (4.3) with (4.2) gives

$$P(A) + P(B) - 1 \le P(A \cap B) \le P(A) + P(B). \tag{4.4}$$

The second inequality in (4.4) is not useful because (4.1) gives a better[3] bound. So, we will replace the second inequality with (4.1):

$$P(A) + P(B) - 1 \le P(A \cap B) \le \min\{P(A), P(B)\}. \tag{4.5}$$

Finally, $P(A \cap B)$ is also a probability and hence it must be between 0 and 1:

$$0 \le P(A \cap B) \le 1 \tag{4.6}$$

Combining (4.6) and (4.5), we have

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B), 1\}.$$

Note that number one at the end of the expression above is not necessary because the two probabilities under minimization can not exceed 1 themselves. In conclusion,

$$\max\{(P(A) + P(B) - 1), 0\} \le P(A \cap B) \le \min\{P(A), P(B)\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ gives the range $\boxed{\left[\frac{1}{6}, \frac{1}{2}\right]}$.

Note that the upper-bound can be obtained by constructing an example which has $A \subset B$. The lower-bound can be obtained by considering an example where $A \cup B = \Omega$.

[2]Again, to see this, note that $A \cap B \subset A$ and $A \cap B \subset B$. Hence, we know that $P(A \cap B) \le P(A)$ and $P(A \cap B) \le P(B)$.

[3]When we already know that a number is less than 3, learning that it is less than 5 does not give us any new information.

(b) We will try to derive general bounds for $P(A \cup B)$.

By monotonicity, because both A and B are subset of $A \cup B$, we must have

$$P(A \cup B) \geq \max\{P(A), P(B)\}.$$

On the other hand, we know, from the finite sub-additivity property, that

$$P(A \cup B) \leq P(A) + P(B).$$

Therefore,

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq P(A) + P(B).$$

Being a probability, $P(A \cup B)$ must be between 0 and 1. Hence,

$$\max\{P(A), P(B), 0\} \leq P(A \cup B) \leq \min\left\{(P(A) + P(B)), 1\right\}.$$

Note that number 0 is not needed in the aximization because the two probabilities involved are automatically ≥ 0 themselves.

In conclusion,

$$\max\{P(A), P(B)\} \leq P(A \cup B) \leq \min\{(P(A) + P(B)), 1\}.$$

Plugging in the value $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$, we have

$$P(A \cup B) \in \boxed{\left[\frac{1}{2}, \frac{5}{6}\right]}.$$

The upper-bound can be obtained by making $A \perp B$. The lower-bound is achieved when $B \subset A$.