ECS 315: Probability and Random Processes **2016/1**

HW Solution 14 — Due: Not Due

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Problem 1. Let a continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the probability density function of X is

$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability that a current measurement is less than 5 milliamperes.

(b) Find and plot the cumulative distribution function of the random variable X.

(c) Find the expected value of X.

(d) Find the variance and the standard deviation of X.

(e) Find the expected value of power when the resistance is 100 ohms?

Solution: See handwritten solution.

Problem 2. The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the probability density function of X.

(b) What proportion of reactions is complete within 200 milliseconds?

Solution: See handwritten solution.

Q1: pdf and cdf - chemical reaction

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$$F_X(x) = \begin{cases} 1 - e^{-0.01x}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $F_X(x)$ is a continuous function. Therefore, X is a continuous RV.

(a) $$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} -(-0.01)e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases} = \begin{cases} 0.01 e^{-0.01x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

At $x = 0$, the derivative does not exist. Because this is just a point, we may assign $f_X(0)$ to be any arbitrary value. Here, we set $f_X(0) = 0$:

$$f_X(x) = \begin{cases} 0.01 e^{-0.01x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $P[X < 200] = P[X \le 200] = F_X(200) = 1 - e^{-0.01 \times 200} = 1 - e^{-2} \approx 0.8647.$

Alternatively, $$P[X < 200] = \int_{-\infty}^{200} f_X(x)\,dx = \underbrace{\cancel{\int_{-\infty}^{0} f_X(x)\,dx}}_{0} + \int_{0}^{200} f_X(x)\,dx$$

$$= \int_0^{200} 0.01 e^{-0.01x}\,dx = \left.\frac{0.01}{(-0.01)} e^{-0.01x}\right|_0^{200}$$

$$= \left(-e^{-0.01 \times 200}\right) - \left(-e^{-0.01 \times 0}\right) = -e^{-2} - (-1)$$

$$= 1 - e^{-2}$$

Q2: pdf, cdf, expected value, variance - current and power

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$$f_X(x) = \begin{cases} 5, & 4.9 \le x \le 5.1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) $$P[X < 5] = \int_{-\infty}^{5} f_X(x)\,dx = \cancelto{0}{\int_{-\infty}^{4.9} \underbrace{f_X(x)}_{0}\,dx} + \int_{4.9}^{5} \underbrace{f_X(x)}_{5}\,dx$$

$$= 5x \Big|_{4.9}^{5} = 5(5-4.9) = 5 \times 0.1 = 0.5$$

(b) $$F_X(x) = P[X \le x] = \int_{-\infty}^{x} f_X(t)\,dt$$

For $x < 4.9$, $f_X(t) = 0$ for all t inside $(0, -\infty)$.

Therefore, $F_X(x) = \int_{-\infty}^{x} 0\,dt = 0.$

For $4.9 \le x \le 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancelto{0}{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt} + \int_{4.9}^{x} \underbrace{f_X(t)}_{5}\,dt$$

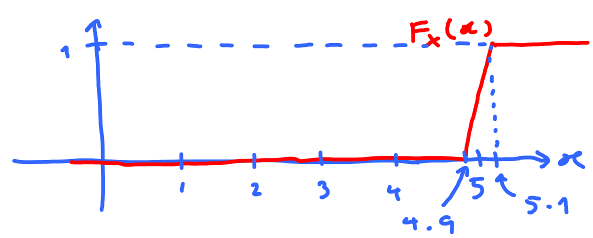
$$= 5t \Big|_{4.9}^{x} = 5 \times (x - 4.9) = 5x - 24.5.$$

For $x > 5.1$, $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \cancelto{0}{\int_{-\infty}^{4.9} \underbrace{f_X(t)}_{0}\,dt} + \int_{4.9}^{5.1} \underbrace{f_X(t)}_{5}\,dt + \cancelto{0}{\int_{5.1}^{x} \underbrace{f_X(t)}_{0}\,dt}$$

$$= 5t \Big|_{4.9}^{5.1} = 5 \times (5.1 - 4.9) = 5 \times 0.2 = 1.$$

Combining the three cases above, we have the complete description of the cdf:

$$F_X(x) = \begin{cases} 0, & x < 4.9, \\ 5x - 24.5, & 4.9 \le x \le 5.1, \\ 1, & x > 5.1 \end{cases}$$



Note that F_X is a continuous function. This is because it is the cdf of a continuous RV.

(c) $\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{0}^{4.9} x \cancel{\underbrace{f_X(x)}_{0}}^{0}\,dx + \int_{4.9}^{5.1} x \underbrace{f_X(x)}_{5}\,dx + \int_{5.1}^{\infty} x \cancel{\underbrace{f_X(x)}_{0}}^{0}\,dx$

$= 5 \left.\frac{x^2}{2}\right|_{4.9}^{5.1} = \frac{5}{2}\left(5.1^2 - 4.9^2\right) = \frac{5}{2}(5.1+4.9)(5.1-4.9) = \frac{5}{2}(10)(0.2)$

$= 5$ mA

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\mathbb{E}X = \frac{b+a}{2} = \frac{5.1+4.9}{2} = \frac{10}{2} = 5.$

(d) $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$. From (c), we know that $\mathbb{E}X = 5$. So, to find $\operatorname{Var} X$, we need to find $\mathbb{E}[X^2]$.

$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{4.9}^{5.1} x^2 \times 5\,dx = 5 \left.\frac{x^3}{3}\right|_{4.9}^{5.1} = \frac{5}{3} \times (5.1^3 - 4.9^3)$

$= 25 + \frac{1}{300}.$

Therefore, $\operatorname{Var} X = \left(25 + \frac{1}{300}\right) - 5^2 = \frac{1}{300} \approx 0.0033\ (\text{mA})^2$

and $\sigma_X = \frac{1}{10\sqrt{3}}$ mA ≈ 0.0577 mA.

Alternatively, for $X \sim \mathcal{U}(a,b)$, we have $\operatorname{Var} X = \frac{(b-a)^2}{12} = \frac{(5.1-4.9)^2}{12}$

$= \frac{(0.2)^2}{12} = \frac{4}{100 \times 12} = \frac{1}{300}.$

(e) Recall that $P = IV = I \times I = I^2 r$.

Here $I = X$. Therefore $P = X^2 r$ and

$\mathbb{E}P = \mathbb{E}[X^2 r] = r\,\mathbb{E}[X^2] = 100 \times \left(25 + \frac{1}{300}\right) = 2500 + \frac{1}{3}$

$\approx 2.50033 \times 10^3 \left[\underbrace{(\text{mA})^2 \Omega}_{m^2 \underbrace{A^2\Omega}_{W}}\right] \approx 2.5$ mW.

Caution: The current is in mA.

Problem 3. Let $X \sim \mathcal{E}(5)$ and $Y = 2/X$.

(a) Check that Y is still a continuous random variable.

(b) Find $F_Y(y)$.

(c) Find $f_Y(y)$.

(d) (optional) Find $\mathbb{E}Y$. Hint: Because $\frac{d}{dy}e^{-\frac{10}{y}} = \frac{10}{y^2}e^{-\frac{10}{y}} > 0$ for $y \neq 0$. We know that $e^{-\frac{10}{y}}$ is an increasing function on our range of integration. In particular, consider $y > 10/\ln(2)$. Then, $e^{-\frac{10}{y}} > \frac{1}{2}$. Hence,

$$\int_0^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}e^{-\frac{10}{y}}dy > \int_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2}dy = \int_{10/\ln 2}^\infty \frac{5}{y}dy.$$

Remark: To be technically correct, we should be a little more careful when writing $Y = \frac{2}{X}$ because it is undefined when $X = 0$. Of course, this happens with 0 probability; so it won't create any serious problem. However, to avoid the problem, we may define Y by

$$Y = \begin{cases} 2/X, & X \neq 0, \\ 0, & X = 0. \end{cases} \tag{14.1}$$

Solution: Here, $X \sim \mathcal{E}(5)$. Therefore, X is a continuous random variable. In this question, we have $Y = g(X)$ where the function g is defined by $g(x) = \frac{2}{x}$.

(a) First, we count the number of solutions for $y = g(x)$.

- For each value of $y > 0$, there is only one x value that satisfies $y = g(x)$. (That x value is $x = \frac{2}{y}$.)
- When y = 0, we need $x = \infty$ or $-\infty$ to make $g(x) = 0$. However, $\pm\infty$ are not real numbers therefore they are not possible x values.
  Note that if we use (14.1), then $x = 0$ is the only solution for $y = g(x)$.
- When $y < 0$, there is no x in the support of X that satisfies $y = g(x)$.

In all three cases, for each value of y, the number of solutions for $y = g(x)$ is (at most) countable. Therefore, because X is a continuous random variable, we conclude that Y is also a continuous random variable.

(b) We consider two cases: "$y \le 0$" and "$y > 0$".

- Because $X > 0$, we know that $Y = \frac{2}{X}$ must be > 0 and hence, $F_Y(y) = 0$ for $y \le 0$.

- For $y > 0$,
$$F_Y(y) = P[Y \le y] = P\left[\frac{2}{X} \le y\right] = P\left[X \ge \frac{2}{y}\right].$$
Note that, for the last equality, we can freely move X and y without worrying about "flipping the inequality" or "division by zero" because both X and y considered here are strictly positive. Now, for $X \sim \mathcal{E}(\lambda)$ and $x > 0$, we have
$$P[X \ge x] = \int\limits_x^\infty \lambda e^{-\lambda t} dt = \left. -e^{-\lambda t}\right|_x^\infty = e^{-\lambda x}$$
Therefore,
$$F_Y(y) = e^{-5\left(\frac{2}{y}\right)} = e^{\frac{-10}{y}}.$$

Combining the two cases above we have
$$F_Y(y) = \boxed{\begin{cases} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0 \end{cases}}$$

(c) Because we have already derived the cdf in the previous part, we can find the pdf via the cdf by $f_Y(y) = \frac{d}{dy} F_Y(y)$. This gives f_Y at all points except at $y = 0$ which we will set f_Y to be 0 there. (This arbitrary assignment works for continuous RV. This is why we need to check first that the random variable is actually continuous.) Hence,
$$f_Y(y) = \boxed{\begin{cases} \frac{10}{y^2} e^{-\frac{10}{y}}, & y > 0 \\ 0, & y \le 0. \end{cases}}$$

(d) We can find $\mathbb{E}Y$ from $f_Y(y)$ found in the previous part or we can even use $f_X(x)$

Method 1:
$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y) = \int\limits_0^\infty y \frac{10}{y^2} e^{-\frac{10}{y}} dy = \int\limits_0^\infty \frac{10}{y} e^{-\frac{10}{y}} dy$$
From the hint, we have
$$\begin{aligned}\mathbb{E}Y &> \int\limits_{10/\ln 2}^\infty \frac{10}{y} e^{-\frac{10}{y}} dy > \int\limits_{10/\ln 2}^\infty \frac{10}{y}\frac{1}{2} dy = \int\limits_{10/\ln 2}^\infty \frac{5}{y} dy \\ &= \left. 5\ln y\right|_{10/\ln 2}^\infty = \infty.\end{aligned}$$

Therefore, $\mathbb{E}Y = \boxed{\infty}$.

Method 2:

$$\mathbb{E}Y = \mathbb{E}\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx = \int_0^{\infty} \frac{1}{x} \lambda e^{-\lambda x} dx > \int_0^1 \frac{1}{x} \lambda e^{-\lambda x} dx$$

$$> \int_0^1 \frac{1}{x} \lambda e^{-\lambda} dx = \lambda e^{-\lambda} \int_0^1 \frac{1}{x} dx = \lambda e^{-\lambda} \ln x\big|_0^1 = \infty,$$

where the second inequality above comes from the fact that for $x \in (0,1)$, $e^{-\lambda x} > e^{-\lambda}$.

Problem 4. In wireless communications systems, fading is sometimes modeled by ***lognormal*** random variables. We say that a positive random variable Y is lognormal if $\ln Y$ is a normal random variable (say, with expected value m and variance σ^2).

Hint: First, recall that the ln is the natural log function (log base e). Let $X = \ln Y$. Then, because Y is lognormal, we know that $X \sim \mathcal{N}(m, \sigma^2)$. Next, write Y as a function of X.

(a) Check that Y is still a continuous random variable.

(b) Find the pdf of Y.

Solution:

Because $X = \ln(Y)$, we have $Y = e^X$. So, here, we consider $Y = g(X)$ where the function g is defined by $g(x) = e^x$.

(a) First, we count the number of solutions for $y = g(x)$. Note that for each value of $y > 0$, there is only one x value that satisfies $y = g(x)$. (That x value is $x = \ln(y)$.) For $y \le 0$, there is no x that satisfies $y = g(x)$. In both cases, the number of solutions for $y = g(x)$ is countable. Therefore, because X is a continuous random variable, we conclude that Y is also a continuous random variable.

(b) Start with $Y = e^X$. We know that exponential function gives strictly positive number. So, Y is always strictly positive. In particular, $F_Y(y) = 0$ for $y \le 0$.

Next, for $y > 0$, by definition, $F_Y(y) = P[Y \le y]$. Plugging in $Y = e^X$, we have

$$F_Y(y) = P\left[e^X \le y\right].$$

Because the exponential function is strictly increasing, the event $[e^X \le y]$ is the same as the event $[X \le \ln y]$. Therefore,

$$F_Y(y) = P[X \le \ln y] = F_X(\ln y).$$

Combining the two cases above, we have

$$F_Y(y) = \begin{cases} F_X(\ln y), & y > 0, \\ 0, & y \le 0. \end{cases}$$

Finally, we apply

$$f_Y(y) = \frac{d}{dy} F_Y(y).$$

For $y < 0$, we have $f_Y(y) = \frac{d}{dy} 0 = 0$. For $y > 0$,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\ln y) = f_X(\ln y) \times \frac{d}{dy} \ln y = \frac{1}{y} f_X(\ln y). \tag{14.2}$$

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & y < 0. \end{cases}$$

At $y = 0$, because Y is a continuous random variable, we can assign any value, e.g. 0, to $f_Y(0)$. Then

$$f_Y(y) = \boxed{\begin{cases} \frac{1}{y} f_X(\ln y), & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Here, $X \sim \mathcal{N}(m, \sigma^2)$. Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

and

$$\boxed{f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{1}{2}\left(\frac{\ln(y)-m}{\sigma}\right)^2}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}}$$

Problem 5. The input X and output Y of a system subject to random perturbations are described probabilistically by the following joint pmf matrix:

$$\begin{array}{c c} x \backslash y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0.02 & 0.10 & 0.08 \\ 0.08 & 0.32 & 0.40 \end{bmatrix} \end{array}$$

(a) Evaluate the following quantities:

(i) The marginal pmf $p_X(x)$

(ii) The marginal pmf $p_Y(y)$

(iii) $\mathbb{E}X$

(iv) $\operatorname{Var} X$

(v) $\mathbb{E}Y$

(vi) $\operatorname{Var} Y$

(vii) $P[XY < 6]$

(viii) $P[X = Y]$

(ix) $\mathbb{E}[XY]$

(x) $\mathbb{E}[(X-3)(Y-2)]$

(xi) $\mathbb{E}[X(Y^3 - 11Y^2 + 38Y)]$

(xii) $\operatorname{Cov}[X, Y]$

(xiii) $\rho_{X,Y}$

(b) Find $\rho_{X,X}$

(c) Calculate the following quantities using the values of $\operatorname{Var} X$, $\operatorname{Cov}[X, Y]$, and $\rho_{X,Y}$ that you got earlier.

(i) $\operatorname{Cov}[3X + 4, 6Y - 7]$

(ii) $\rho_{3X+4,6Y-7}$

(iii) $\operatorname{Cov}[X, 6X - 7]$

(iv) $\rho_{X,6X-7}$

Solution:

(a) The `MATLAB` codes are provided in the file `P_XY_EVarCov.m`.

(i) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \boxed{\begin{cases} 0.2, & x = 1 \\ 0.8, & x = 3 \\ 0, & \text{otherwise.} \end{cases}}$$

(ii) The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \boxed{\begin{cases} 0.1, & y = 2 \\ 0.42, & y = 4 \\ 0.48, & y = 5 \\ 0, & \text{otherwise.} \end{cases}}$$

(iii) $\mathbb{E}X = \sum_x x p_X(x) = 1 \times 0.2 + 3 \times 0.8 = 0.2 + 2.4 = \boxed{2.6}$.

(iv) $\mathbb{E}[X^2] = \sum_x x^2 p_X(x) = 1^2 \times 0.2 + 3^2 \times 0.8 = 0.2 + 7.2 = 7.4$.

So, $\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2 = 7.4 - (2.6)^2 = 7.4 - 6.76 = \boxed{0.64}$.

(v) $\mathbb{E}Y = \sum_y y p_Y(y) = 2 \times 0.1 + 4 \times 0.42 + 5 \times 0.48 = 0.2 + 1.68 + 2.4 = \boxed{4.28}$.

(vi) $\mathbb{E}[Y^2] = \sum_y y^2 p_Y(y) = 2^2 \times 0.1 + 4^2 \times 0.42 + 5^2 \times 0.48 = 19.12$.

So, $\operatorname{Var} Y = \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 = 19.12 - 4.28^2 = \boxed{0.8016}$.

(vii) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, only the pairs $(1,2)$, $(1,4)$, $(1,5)$ satisfy $xy < 6$. Therefore, $[XY < 6] = [X = 1]$ which implies $P[XY < 6] = P[X = 1] = \boxed{0.2}$.

(viii) Among the 6 possible pairs of (x, y) shown in the joint pmf matrix, there is no pair which has $x = y$. Therefore, $P[X = Y] = \boxed{0}$.

(ix) First, we calculate the values of $x \times y$:

$$\begin{array}{cc} x \setminus y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 2 & 4 & 5 \\ 6 & 12 & 15 \end{bmatrix} \end{array}$$

Then, each $x \times y$ is weighted (multiplied) by the corresponding probability $p_{X,Y}(x, y)$:

$$\begin{array}{cc} x \setminus y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0.04 & 0.40 & 0.40 \\ 0.48 & 3.84 & 6.00 \end{bmatrix} \end{array}$$

Finally, $\mathbb{E}[XY]$ is sum of these numbers. Therefore, $\mathbb{E}[XY] = \boxed{11.16}$.

(x) First, we calculate the values of $(x - 3) \times (y - 2)$:

$$\begin{array}{cc} x \setminus y & \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \begin{bmatrix} 0 & -4 & -6 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Then, each $(x-3)\times(y-2)$ is weighted (multiplied) by the corresponding probability $p_{X,Y}(x,y)$:

$$\begin{array}{cc|ccc} & y-2 & 0 & 2 & 3 \\ x-3 & x\setminus y & 2 & 4 & 5 \\ -2 & 1 & \multicolumn{3}{c}{\multirow{2}{*}{$\begin{bmatrix} 0 & -0.40 & -0.48 \\ 0 & 0 & 0 \end{bmatrix}$}} \\ 0 & 3 & & & \end{array}$$

Finally, $\mathbb{E}\left[(X-3)(Y-2)\right]$ is sum of these numbers. Therefore,

$$\mathbb{E}\left[(X-3)(Y-2)\right] = \boxed{-0.88}.$$

(xi) First, we calculate the values of $x(y^3-11y^2+38y)$:

$$\begin{array}{l|ccc} y^3-11y^2+38y & 40 & 40 & 40 \\ x\setminus y & 2 & 4 & 5 \\ 1 & 40 & 40 & 40 \\ 3 & 120 & 120 & 120 \end{array}$$

Then, each $x(y^3-11y^2+38y)$ is weighted (multiplied) by the corresponding probability $p_{X,Y}(x,y)$:

$$\begin{array}{c|ccc} x\setminus y & 2 & 4 & 5 \\ 1 & 0.8 & 4.0 & 3.2 \\ 3 & 9.6 & 38.4 & 48.0 \end{array}$$

Finally, $\mathbb{E}\left[X(Y^3-11Y^2+38Y)\right]$ is sum of these numbers. Therefore,

$$\mathbb{E}\left[X(Y^3-11Y^2+38Y)\right] = \boxed{104}.$$

(xii) $\operatorname{Cov}[X,Y] = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y = 11.16 - (2.6)(4.28) = \boxed{0.032}$.

(xiii) $\rho_{X,Y} = \frac{\operatorname{Cov}[X,Y]}{\sigma_X\sigma_Y} = \frac{0.032}{\sqrt{0.64}\sqrt{0.8016}} = \boxed{0.044677}$

(b) $\rho_{X,X} = \frac{\operatorname{Cov}[X,X]}{\sigma_X\sigma_X} = \frac{\operatorname{Var}[X]}{\sigma_X^2} = \boxed{1}$.

(c)

(i) $\operatorname{Cov}[3X+4, 6Y-7] = 3\times 6\times \operatorname{Cov}[X,Y] \approx 3\times 6\times 0.032 \approx \boxed{0.576}$.

(ii) Note that

$$\begin{aligned}\rho_{aX+b,cY+d} &= \frac{\operatorname{Cov}[aX+b,cY+d]}{\sigma_{aX+b}\sigma_{cY+d}} \\ &= \frac{ac\operatorname{Cov}[X,Y]}{|a|\sigma_X|c|\sigma_Y} = \frac{ac}{|ac|}\rho_{X,Y} = \operatorname{sign}(ac)\times\rho_{X,Y}.\end{aligned}$$

Hence, $\rho_{3X+4,6Y-7} = \text{sign}(3 \times 4)\rho_{X,Y} = \rho_{X,Y} = \boxed{0.0447}$.

(iii) $\text{Cov}[X, 6X - 7] = 1 \times 6 \times \text{Cov}[X, X] = 6 \times \text{Var}[X] \approx \boxed{3.84}$.

(iv) $\rho_{X,6X-7} = \text{sign}(1 \times 6) \times \rho_{X,X} = \boxed{1}$.

Problem 6. Suppose $X \sim \text{binomial}(5, 1/3)$, $Y \sim \text{binomial}(7, 4/5)$, and $X \perp\!\!\!\perp Y$. Evaluate the following quantities.

(a) $\mathbb{E}[(X - 3)(Y - 2)]$

(b) $\text{Cov}[X, Y]$

(c) $\rho_{X,Y}$

Solution:

(a) First, because X and Y are independent, we have $\mathbb{E}[(X-3)(Y-2)] = \mathbb{E}[X-3]\,\mathbb{E}[Y-2]$. Recall that $\mathbb{E}[aX + b] = a\mathbb{E}[X]+b$. Therefore, $\mathbb{E}[X-3]\,\mathbb{E}[Y-2] = (\mathbb{E}[X] - 3)(\mathbb{E}[Y] - 2)$ Now, for Binomial(n, p), the expected value is np. So,

$$(\mathbb{E}[X] - 3)(\mathbb{E}[Y] - 2) = \left(5 \times \frac{1}{3} - 3\right)\left(7 \times \frac{4}{5} - 2\right) = -\frac{4}{3} \times \frac{18}{5} = \boxed{-\frac{24}{5}} = -4.8.$$

(b) $\text{Cov}[X, Y] = \boxed{0}$ because $X \perp\!\!\!\perp Y$.

(c) $\rho_{X,Y} = \boxed{0}$ because $\text{Cov}[X, Y] = 0$

Extra Questions

Here are some extra questions for those who want more practice.

Problem 7. Consider a random variable X whose pdf is given by

$$f_X(x) = \begin{cases} cx^2, & x \in (1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = 4|X - 1.5|$.

(a) Find $\mathbb{E}Y$.

(b) Find $f_Y(y)$.

Solution: See handwritten solution

Q7: SISO

Saturday, November 22, 2014 11:54 PM

First, we need to find the constant c.

For any pdf, we know that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

Therefore, $\int_1^2 c x^2 dx = c\int_1^2 x^2 dx = c\left.\frac{x^3}{3}\right|_1^2 = c\left(\frac{8-1}{3}\right) = c\times\frac{7}{3}$ must $= 1$.

Hence, $c = 3/7$.

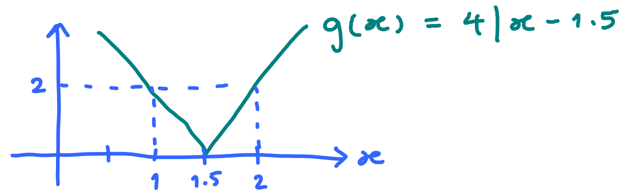
(a) $\mathbb{E}Y = \mathbb{E}\left[4|X-1.5|\right] = 4\int_1^2 |x-1.5|\,\frac{3}{7}x^2 dx = \frac{12}{7}\int_1^2 |x-1.5|\,x^2 dx$

$$|x-1.5| = \begin{cases} x-1.5, & x \geq 1.5 \\ 1.5-x, & x < 1.5 \end{cases}$$

$$= \frac{12}{7}\left(\int_1^{1.5}(1.5-x)\,x^2 dx + \int_{1.5}^{2}(x-1.5)\,x^2 dx\right) = \frac{57}{56}$$

(b) $Y = 4|X-1.5| = \begin{cases} 4X-6, & X \geq 1.5, \\ 6-4X, & X < 1.5 \end{cases} \equiv g(X)$

Let's plot the function $g(x)$:



First, let's check that Y is a cont. RV. This is easy to see from $g(x)$.

For each value of y, there are $\underbrace{\text{at most two}}_{\Downarrow}$ values of x that satisfy $y = g(x)$.

finite $\Rightarrow$ countable $\xRightarrow{X \text{ is a cont. RV}}$ $P[Y=y] = 0\ \forall y$

$\Rightarrow Y$ is a cont. RV.

Step ① : Find the cdf. Step ②: $f_Y(y) = \frac{d}{dy}F_Y(y)$

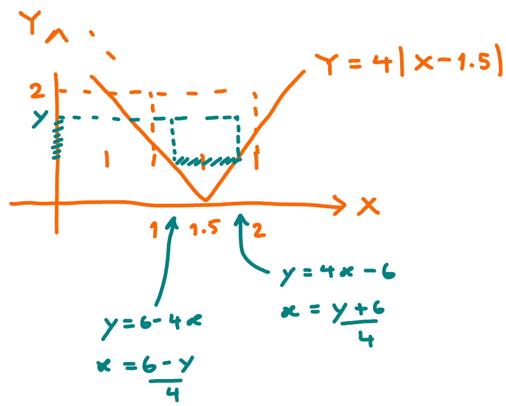
1.1 By construction (from 1.1), we know that $Y \geq 0$. Therefore,

$F_Y(y) = 0$ for $y < 0$.

2.1 This means $f_Y(y) = 0$ for $y < 0$. (*)

1.2 For $y = 0$, $F_Y(0) = P[Y \leq 0] = P[X = 0] \overset{\text{for cont. X.}}{=} 0$ (**)

(1.3) For $y > 0$,



the event $[Y \le y]$ is the same as the event $\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right]$.

Therefore,

for cont. X

$$F_Y(y) = P\left[\frac{6-y}{4} \le X \le \frac{6+y}{4}\right] = F_X\left(\frac{6+y}{4}\right) - F_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0.$$

(2.3) This implies

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{4}f_X\left(\frac{6+y}{4}\right) + \frac{1}{4}f_X\left(\frac{6-y}{4}\right) \quad \text{when } y > 0. \quad (***)$$

Plug-in $f_X(\cdot) = \frac{3}{7}(\cdot)^2$ when $1 < (\cdot) < 2$.

$1 < \frac{6+y}{4} < 2$

$4 < 6+y < 8$

$-2 < y < 2$

$1 < \frac{6-y}{4} < 2$

$4 < 6-y < 8$

$-2 < -y < 2$

$-2 < y < 2$

Note again that this analysis is valid only for $y > 0$.

Therefore,

$$f_Y(y) = \begin{cases} \frac{1}{4}\times\frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2 \\ 0, & y \ge 2 \end{cases}$$

Combining (2.1) and (2.3), we have

$$f_Y(y) = \begin{cases} \frac{1}{4}\times\frac{3}{7}\left(\left(\frac{6+y}{4}\right)^2 + \left(\frac{6-y}{4}\right)^2\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

At $y = 0$, we set $f_Y(0) = 0$. This is possible because Y is a continuous RV.

$$= \begin{cases} \frac{3}{224}\left(y^2 + 36\right), & 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Check $\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 \frac{3}{224}\left(y^3 + 36y\right)dy = \frac{57}{56}$ ← same as part (a).

Problem 8. A webpage server can handle r requests per day. Find the probability that the server gets more than r requests at least once in n days. Assume that the number of requests on day i is $X_i \sim \mathcal{P}(\alpha)$ and that $X_1, \dots, X_n$ are independent.

Solution: [Gubner, 2006, Ex 2.10]

$$P\left[\bigcup_{i=1}^{n} [X_i > r]\right] = 1 - P\left[\bigcap_{i=1}^{n} [X_i \le r]\right] = 1 - \prod_{i=1}^{n} P\left[X_i \le r\right]$$
$$= 1 - \prod_{i=1}^{n}\left(\sum_{k=0}^{r} \frac{\alpha^k e^{-\alpha}}{k!}\right) = \boxed{1 - \left(\sum_{k=0}^{r} \frac{\alpha^k e^{-\alpha}}{k!}\right)^n}.$$

Problem 9. Suppose $X \sim$ binomial$(5, 1/3)$, $Y \sim$ binomial$(7, 4/5)$, and $X \perp\!\!\!\perp Y$.

(a) A vector describing the pmf of X can be created by the MATLAB expression:

```
x = 0:5; pX = binopdf(x,5,1/3).
```

What is the expression that would give `pY`, a corresponding vector describing the pmf of Y?

(b) Use `pX` and `pY` from part (a), how can you create the joint pmf matrix in MATLAB? Do not use "for-loop", "while-loop", "if statement". Hint: Multiply them in an appropriate orientation.

(c) Use `MATLAB` to evaluate the following quantities. Again, do not use "for-loop", "while-loop", "if statement".

(i) $\mathbb{E}X$

(ii) $P[X = Y]$

(iii) $P[XY < 6]$

Solution: The `MATLAB` codes are provided in the file `P_XY_jointfromMarginal_indp.m`.

(a) `y = 0:7; pY = binopdf(y,7,4/5);`

(b) `P = pX.'*pY;`

(c)

(i) $\mathbb{E}X = \boxed{1.667}$

(ii) $P[X = Y] = \boxed{0.0121}$

(iii) $P[XY < 6] = \boxed{0.2727}$

Problem 10. Suppose $\operatorname{Var} X = 5$. Find $\operatorname{Cov}[X, X]$ and $\rho_{X,X}$.
Solution:

(a) $\operatorname{Cov}[X, X] = \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)] = \mathbb{E}[(X - \mathbb{E}X)^2] = \operatorname{Var} X = \boxed{5}$.

(b) $\rho_{X,X} = \frac{\operatorname{Cov}[X,X]}{\sigma_X \sigma_X} = \frac{\operatorname{Var} X}{\sigma_X^2} = \frac{\operatorname{Var} X}{\operatorname{Var} X} = \boxed{1}$.

Problem 11. Suppose we know that $\sigma_X = \frac{\sqrt{21}}{10}$, $\sigma_Y = \frac{4\sqrt{6}}{5}$, $\rho_{X,Y} = -\frac{1}{\sqrt{126}}$.

(a) Find $\operatorname{Var}[X + Y]$.

(b) Find $\mathbb{E}[(Y - 3X + 5)^2]$. Assume $\mathbb{E}[Y - 3X + 5] = 1$.

Solution:

(a) First, we know that $\operatorname{Var} X = \sigma_X^2 = \frac{21}{100}$, $\operatorname{Var} Y = \sigma_Y^2 = \frac{96}{25}$, and $\operatorname{Cov}[X, Y] = \rho_{X,Y} \times \sigma_X \times \sigma_Y = -\frac{2}{25}$. Now,

$$
\begin{aligned}
\operatorname{Var}[X+Y] &= \mathbb{E}\left[((X+Y) - \mathbb{E}[X+Y])^2\right] = \mathbb{E}\left[((X - \mathbb{E}X) + (Y - \mathbb{E}Y))^2\right] \\
&= \mathbb{E}\left[(X - \mathbb{E}X)^2\right] + 2\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] + \mathbb{E}\left[(Y - \mathbb{E}Y)^2\right] \\
&= \operatorname{Var} X + 2\operatorname{Cov}[X, Y] + \operatorname{Var} Y \\
&= \boxed{\frac{389}{100}} = 3.89.
\end{aligned}
$$

Remark: It is useful to remember that

$$\operatorname{Var}[X + Y] = \operatorname{Var} X + 2\operatorname{Cov}[X, Y] + \operatorname{Var} Y.$$

Note that when X and Y are uncorrelated, $\operatorname{Var}[X + Y] = \operatorname{Var} X + \operatorname{Var} Y$. This simpler formula also holds when X and Y are independence because independence is a stronger condition.

(b) First, we write

$$Y - aX - b = (Y - \mathbb{E}Y) - a(X - \mathbb{E}X) - \underbrace{(a\mathbb{E}X + b - \mathbb{E}Y)}_{c}.$$

Now, using the expansion

$$(u + v + t)^2 = u^2 + v^2 + t^2 + 2uv + 2ut + 2vt,$$

we have

$$\begin{aligned}(Y - aX - b)^2 = (Y - \mathbb{E}Y)^2 + a^2(X - \mathbb{E}X)^2 + c^2 \\ - 2a\,(X - \mathbb{E}X)\,(Y - \mathbb{E}Y) - 2c\,(Y - \mathbb{E}Y) + 2a\,(X - \mathbb{E}X)\,c.\end{aligned}$$

Recall that $\mathbb{E}\left[X - \mathbb{E}X\right] = \mathbb{E}\left[Y - \mathbb{E}Y\right] = 0$. Therefore,

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + c^2 - 2a\mathrm{Cov}\left[X, Y\right]$$

Plugging back the value of c, we have

$$\boxed{\mathbb{E}\left[(Y - aX - b)^2\right] = \operatorname{Var} Y + a^2 \operatorname{Var} X + (\mathbb{E}\left[(Y - aX - b)\right])^2 - 2a\mathrm{Cov}\left[X, Y\right]}.$$

Here, $a = 3$ and $b = -5$. Plugging these values along with the given quantities into the formula gives

$$\mathbb{E}\left[(Y - aX - b)^2\right] = \boxed{\frac{721}{100}} = 7.21.$$

Problem 12. The input X and output Y of a system subject to random perturbations are described probabilistically by the joint pmf $p_{X,Y}(x, y)$, where $x = 1, 2, 3$ and $y = 1, 2, 3, 4, 5$. Let P denote the joint pmf matrix whose i,j entry is $p_{X,Y}(i, j)$, and suppose that

$$P = \frac{1}{71}\begin{bmatrix} 7 & 2 & 8 & 5 & 4 \\ 4 & 2 & 5 & 5 & 9 \\ 2 & 4 & 8 & 5 & 1 \end{bmatrix}$$

(a) Find the marginal pmfs $p_X(x)$ and $p_Y(y)$.

(b) Find $\mathbb{E}X$

(c) Find $\mathbb{E}Y$

(d) Find $\operatorname{Var} X$

(e) Find $\operatorname{Var} Y$

Solution: All of the calculations in this question are simply plugging numbers into appropriate formula. The `MATLAB` codes are provided in the file `P_XY_marginal_2.m`.

(a) The marginal pmf $p_X(x)$ is founded by the sums along the rows of the pmf matrix:

$$p_X(x) = \begin{cases} 26/71, & x = 1 \\ 25/71, & x = 2 \\ 20/71, & x = 3 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.3662, & x = 1 \\ 0.3521, & x = 2 \\ 0.2817, & x = 3 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pmf $p_Y(y)$ is founded by the sums along the columns of the pmf matrix:

$$p_Y(y) = \begin{cases} 13/71, & y = 1 \\ 8/71, & y = 2 \\ 21/71, & y = 3 \\ 15/71, & y = 4 \\ 14/71, & y = 5 \\ 0, & \text{otherwise} \end{cases} \approx \begin{cases} 0.1831, & y = 1 \\ 0.1127, & y = 2 \\ 0.2958, & y = 3 \\ 0.2113, & y = 4 \\ 0.1972, & y = 5 \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\mathbb{E}X = \frac{136}{71} \approx 1.9155$

(c) $\mathbb{E}Y = \frac{222}{71} \approx 3.1268$

(d) $\operatorname{Var} X = \frac{3230}{5041} \approx 0.6407$

(e) $\operatorname{Var} Y = \frac{9220}{5041} \approx 1.8290$